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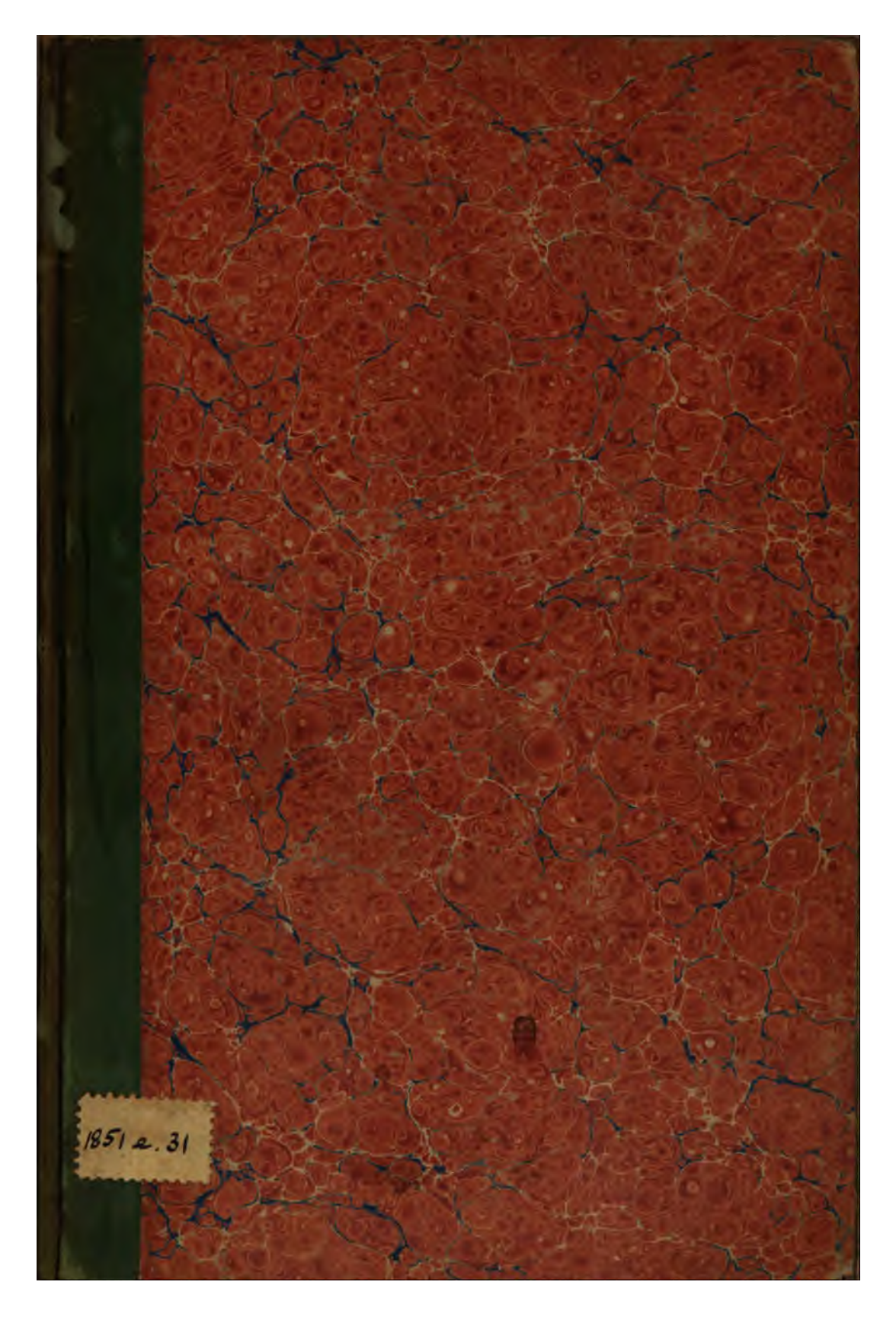
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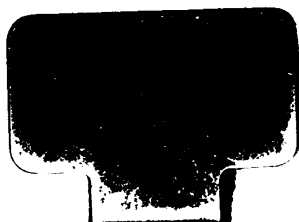
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OPTICAL PROBLEMS.

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PREFACE.

THE want which has been felt within the last few years of a systematic collection of Examples, elucidating the various theorems of Geometrical Optics, has induced me to publish the following pages. I have endeavoured, as far as the narrow limits that I had assigned myself allowed, to give each branch of the subject a fair amount of illustration. There is so little scope for originality in this department of Natural Philosophy, that a short treatise presenting, in their most general form, a few problems of each species will be found sufficient in the present state of the science. I have adapted the arrangement as much as possible to the method pursued in Griffin's *Treatise on Geometrical Optics*, and I hope that my collection will be found a useful supplement to it.

Craig by Kilmarnock, April 1850.

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OPTICAL PROBLEMS.

CHAPTER I.

ILLUMINATION.

If a luminous surface (A) illuminates a surface (B), we find the illumination from A at a particular point of B by the method given in Griffin's *Optics* (Appendix, Art. 225), and then the illumination on an element of B at that point. We have now to integrate throughout the extent of B which is illuminated.

If a sphere be described with the point P of the illuminated surface as centre and radius R , and if (a) be the area of this sphere, intersected by a line through P moving round the boundary of A , it is shewn that the illumination at P from

$$A = \frac{C}{R^2} \times \text{area of projection of } (a) \text{ on the plane of } P.$$

1. A plane surface touches a self-luminous paraboloid of revolution at its vertex, and is then moved parallel to itself along the axis produced; prove that the illumination of the plane at the point where the axis meets it, varies inversely as its distance from the focus.

Let (a) (fig. 1) be the vertex of the paraboloid; AB and AC tangents from A , defining the portion of it from which A receives illumination.

Also let S be the focus, $4a$ the latus rectum, $CN = y$, $aN = x = Aa$ by a property of the parabola.

Then if a sphere be described with centre A and radius AB or AC , and BOC be the surface intersected, illumination at A

$$\begin{aligned}
 &= \frac{C}{AC^2} \times \text{area of projection of } BOC, \\
 &= \frac{C \cdot \pi \cdot CN^2}{CN^2 + AN^2}, \\
 &= \frac{C \cdot \pi \cdot 4ax}{4ax + 4x^2}, \\
 &= \frac{C \cdot \pi \cdot a}{a + x} = \frac{C \cdot \pi \cdot a}{AS},
 \end{aligned}$$

therefore $\propto \frac{1}{AS}$.

2. Find the illumination at a point of a plane surface, illuminated by a rectangle of infinite length, perpendicular to it.

Let A (fig. 2) be a point of the illuminated area, and EF the base of the rectangle.

Describe a sphere about A as centre, with any radius AB . A line through A , moving round the boundary of the rectangle, will intersect it in DBC .

Then if $EF = a$, $AE = b$, $AF = c$, illumination at A from the rectangle

$$\begin{aligned}
 &= \frac{C}{AB^2} \times \text{area } BAC, \\
 &= C \cdot \frac{\frac{1}{2}(\text{angle } BAC \cdot AB^2)}{AB^2}, \\
 &= C \frac{1}{2} \cos^{-1} \frac{b^2 + c^2 - a^2}{2bc}.
 \end{aligned}$$

3. Two semi-circular, self-luminous plates are placed with their diameters upon a plane, and with their planes perpendicular to the line which joins their centres A, B ; find that point in the line AB , where the illumination of the plane, upon which the semi-circles are placed, is least.

Let AED , BGC (fig. 3) be the two semi-circles, P the point required. With centre P and radius PA describe a sphere, and let a straight line through P , moving round DEA , intersect it in Apq .

Let $AB = a$, $AP = x$, $AD = r$, $BC = s$, and let Aoq be the projection of Apq .

Then illumination at P from semi-circle at A

$$\begin{aligned} &= C \frac{\text{area of projection } 2Aoq}{AP^2}, \\ &= C \frac{x^2 \tan^{-1} \frac{r}{x} - x^2 \frac{rx}{r^2 + x^2}}{x^3}, \\ &= C \left(\tan^{-1} \frac{r}{x} - \frac{rx}{r^2 + x^2} \right). \end{aligned}$$

Similarly, illumination at P from semi-circle at B

$$= C \left\{ \tan^{-1} \frac{s}{a-x} - \frac{s(a-x)}{s^2 + (a-x)^2} \right\};$$

therefore whole illumination at P

$$= C \left\{ \tan^{-1} \frac{r}{x} + \tan^{-1} \frac{s}{a-x} - \frac{rx}{r^2 + x^2} - \frac{s(a-x)}{s^2 + (a-x)^2} \right\};$$

equating the differential coefficient to zero, we have

$$0 = \frac{-r}{r^2 + x^2} + \frac{s}{s^2 + (a-x)^2} - \frac{r(r^2 - x^2)}{(r^2 + x^2)^2} + \frac{s\{s^2 - (a-x)^2\}}{\{s^2 + (a-x)^2\}^2},$$

$$0 = \frac{-2r^3}{(r^2 + x^2)^2} + \frac{2s^3}{\{s^2 + (a-x)^2\}^2},$$

$$0 = s^3(r^2 + x^2)^2 - r^3\{s^2 + (a-x)^2\}^2;$$

$$\therefore x^2 + \frac{2ar^{\frac{3}{2}}}{s^{\frac{3}{2}} - r^{\frac{3}{2}}} = \frac{(a^2 + s^2)r^{\frac{3}{2}} - r^2 s^{\frac{3}{2}}}{s^{\frac{3}{2}} - r^{\frac{3}{2}}},$$

$$x + \frac{ar^{\frac{1}{2}}}{s^{\frac{3}{2}} - r^{\frac{3}{2}}} = \pm \sqrt{\left\{ \frac{(a^2 + s^2)r^{\frac{3}{2}} - r^2 s^{\frac{3}{2}}}{s^{\frac{3}{2}} - r^{\frac{3}{2}}} + \frac{a^2 r^3}{(s^{\frac{3}{2}} - r^{\frac{3}{2}})^2} \right\}},$$

therefore

$$x = \frac{-ar^{\frac{3}{2}} \pm s^{\frac{3}{2}}r^{\frac{1}{2}}\sqrt{\{(s + r^{\frac{1}{2}}s^{\frac{1}{2}} + r)(s^{\frac{1}{2}} - r^{\frac{1}{2}})^2 + a^2\}}}{s^{\frac{3}{2}} - r^{\frac{3}{2}}};$$

hence the position of P is known.

In this problem, if $r = s$, we have

$$\begin{aligned} x &= \frac{-ar^{\frac{3}{2}} \pm as^{\frac{3}{2}}r^{\frac{1}{2}}}{s^{\frac{3}{2}} - r^{\frac{3}{2}}}, \\ &= ar^{\frac{3}{2}} \frac{-r^{\frac{1}{2}} \pm s^{\frac{1}{2}}}{s^{\frac{3}{2}} - r^{\frac{3}{2}}}, \\ &= \infty \text{ or } ar^{\frac{3}{2}} \frac{1}{2r^{\frac{3}{2}}} = \frac{a}{2}. \end{aligned}$$

In that case we see that P is the bisection of AB .

4. If α be the angle which every diameter of a circular disc subtends at a luminous point, shew that the ratio of the light which falls on the disc, to the whole light emitted, is as $\sin^2 \frac{\alpha}{4} : 1$.

Let A (fig. 4) be the luminous point, EF a diameter, B its middle point. Also $PAB = \theta$, $PB = r$, $AP = \rho$, $AB = a$.

Illumination at P from $A = C \frac{\cos \theta}{\rho^2}$. Now

$$\begin{aligned} r &= a \tan \theta, \\ \rho &= a \sec \theta, \\ dr &= a \sec^2 \theta d\theta; \end{aligned}$$

and illumination upon an annulus round B , of radius r and thickness δr ,

$$\begin{aligned} &= 2\pi C \frac{r \delta r \cos \theta}{\rho^2}, \\ &= 2\pi C \frac{\cos \theta a^2 \tan \theta \sec^2 \theta \delta \theta}{a^2 \sec^3 \theta}, \\ &= 2\pi C \sin \theta d\theta; \end{aligned}$$

$$\begin{aligned}\text{therefore illumination on disc} &= 2\pi C \int_0^{\frac{\alpha}{2}} \sin \theta \, d\theta, \\ &= 4\pi C \sin^2 \frac{\alpha}{4}.\end{aligned}$$

Making $\alpha = 2\pi$, we have: whole light emitted
 $= 4\pi C$;

therefore light which falls on the disc: whole light emitted
 $\therefore \sin^2 \frac{\alpha}{4} : 1$.

5. A luminous point is placed in a hemisphere, in that radius which is perpendicular to the base; compare the illuminations of the curved surface and the base.

Let A (fig. 5) be the luminous point, B the centre of the base, and $EAF = \alpha$; then, as in the preceding problem, the illumination of the base

$$= 4\pi C \sin^2 \frac{\alpha}{4}.$$

Now let P be a point of the curved surface,

$$BAP = \theta, \quad BAQ = \theta + \delta\theta, \quad AP = \rho.$$

Then the area of the annulus formed by the revolution of PQ about the diameter BC

$$= 2\pi\rho \sin \theta \, PQ,$$

$$\text{and} \quad \cos APB = \sin APQ = \frac{\rho \delta\theta}{PQ};$$

therefore illumination of annulus

$$\begin{aligned}&= C \frac{2\pi\rho \sin \theta \, \rho \delta\theta}{\rho^3}, \\ &= 2\pi C \sin \theta \delta\theta,\end{aligned}$$

and illumination of curved surface

$$\begin{aligned}&= 2\pi C \int_{\pi-\frac{\alpha}{2}}^{\pi} \sin \theta \, d\theta, \\ &= 2\pi C \left(1 + \cos \frac{\alpha}{2} \right);\end{aligned}$$

therefore

$$\begin{aligned} \frac{\text{illumination of base}}{\text{illumination of curved surface}} &= \frac{2 \sin^2 \frac{a}{4}}{1 + \cos \frac{a}{2}}, \\ &= \frac{1 - \cos \frac{a}{2}}{1 + \cos \frac{a}{2}}, \\ &= \tan^2 \frac{a}{4}. \end{aligned}$$

6. Find the quantity of light received by an infinite plate placed at a given distance from a self-luminous sphere.

Let A (fig. 6) be the centre of the illuminating sphere, AB perpendicular on the plane, P a point of the plane, QPq the enveloping cone, with vertex P , touching the sphere in the circle Qq .

Then illumination at $P = \frac{C}{AP^2} \times \text{projection of } Qq$

$$= \frac{C}{AP^2} \pi PQ^2 \sin^2 QPA \sin APB.$$

Now let $AB = a$, $AQ = b$, $BP = r$,
therefore $AP^2 = a^2 + r^2$,

$$PQ^2 = AP^2 - b^2 = a^2 - b^2 + r^2,$$

$$\sin QPA = \frac{b}{AP}, \quad \sin APB = \frac{a}{AP};$$

therefore illumination at $P = \pi C \frac{ab^2 \cdot PQ^2}{AP^5},$

$$= \pi C ab^2 \frac{a^2 - b^2 + r^2}{(a^2 + b^2)^{\frac{5}{2}}}.$$

If we take an annulus through P , around B as centre, of breadth δr , illumination theorem

$$= 2\pi^2 C ab^2 \frac{a^2 - b^2 + r^2}{(a^2 + b^2)^{\frac{5}{2}}} r \delta r,$$

$$\text{therefore illumination required} = 2\pi^2 Cab^3 \int_0^\infty \frac{a^2 - b^2 + r^2}{(a^2 + b^2)^{\frac{5}{2}}} r dr.$$

$$\text{Let} \quad \sqrt{(a^2 + r^2)} = z,$$

$$\text{therefore} \quad \frac{r dr}{\sqrt{(a^2 + r^2)}} = dz,$$

$$\begin{aligned} \text{therefore illumination} &= 2\pi^2 Cab^3 \int_0^\infty \frac{(z^2 - b^2) dz}{z^4}, \\ &= 2\pi^2 Cab^3 \left(\frac{1}{a} - \frac{b^3}{3a^3} \right). \end{aligned}$$

CHAPTER II.

DIRECT REFLECTION AND REFRACTION.

1. A plane mirror, moveable about an axis in its own plane, parallel to the axis of the earth, revolves from east to west, with half the sun's apparent diurnal motion. Shew that the direction of the reflected rays of sun-light will not be sensibly altered during the day.

Let $\triangle AOP$ (fig. 7) be the position of the mirror when the sun is at S ; $\triangle A'OP$ and S' two other corresponding positions; then angle $SOS' = 2\angle AOA$. Draw OB, OB' normals to $\triangle AOP, \triangle A'OP$. Then, if OC be the direction of the reflected rays in the first position,

$$BOC = BOS.$$

$$\begin{aligned} \text{Now} \quad B'OS' &= BOS - \frac{SOS'}{2}, \\ &= BOS - BOB', \\ &= BOC - BOB', \\ &= B'OC; \end{aligned}$$

therefore OC is the direction of the reflected rays.

2. A ray of light is incident from a point P , and reflected at a given plane surface to a fixed point; if the whole length of the incident and reflected ray be proportional to the distance between the points, the locus of P is a sphere.

Let Q be the given fixed point; draw QO perpendicular upon the given plane, and take O for origin of coordinates, and OQ for the axis of x . Then, if x, y, z , be the coordinates of P , and PrQ the course of the reflected ray,

$$Pr + Qr = \mu PQ.$$

Let OP' (fig. 8) be the projection of QP upon the plane yz , and $OQ = a$; then

$$\frac{PP'}{Pr} = \frac{QO}{Qr}, \quad \therefore \frac{x}{Pr} = \frac{a}{Qr},$$

$$\frac{x}{a} = \frac{Pr}{Qr}, \quad \frac{x+a}{a} = \frac{\mu PQ}{Qr};$$

therefore $\frac{x+a}{a} \sqrt{(a^2 + Or^2)} = \mu \sqrt{\{y^2 + z^2 + (a-x)^2\}}.$

Now $\frac{Or}{P'r} = \frac{a}{x}, \quad \therefore \frac{Or}{OP'} = \frac{a}{a+x},$

and therefore $Or = \frac{a}{a+x} \sqrt{(y^2 + z^2)};$

$$\therefore \frac{x+a}{a} \sqrt{\left\{a^2 + \frac{a^2}{(a+x)^2} (y^2 + z^2)\right\}} = \mu \sqrt{\{y^2 + z^2 + (a-x)^2\}},$$

$$a^2(a+x)^2 + a^2(y^2 + z^2) = a^2\mu^2 \{y^2 + z^2 + (a-x)^2\}.$$

Hence $x^2 + y^2 + z^2 = \frac{\mu^2 + 1}{\mu^2 - 1} 2ax - a^2,$

which is the equation to a sphere whose centre is in the axis of x .

3. In a parabolic reflector a ray of light is incident from a luminous point in the axis, so that the reflected ray cuts the axis at right angles; prove that the angle between the incident ray and the focal distance of the point of incidence is a right angle. Given the distance between the luminous point and the focus to be twice the latus rectum, find where the incident ray cuts the axis.

Let Q (fig. 9) be the luminous point in the axis, A the vertex, and S the focus of the parabolic reflector, PT a tangent, and PG a normal at P . Then if

$$QPG = \theta = NPG,$$

$$PQG = PGS - \theta,$$

$$= SPG - \theta,$$

$$= SPN.$$

But PQG also $= 90^\circ - 2\theta$, therefore

$$SPN = 90^\circ - 2\theta, \text{ or } SPN + NPQ = 90^\circ.$$

Again, if SQ be given $= 8AS$,

$$AQ = 9AS;$$

and by similar triangles SPN, SPQ , we have

$$\frac{SN}{SP} = \frac{SP}{SQ},$$

$$\text{therefore } SN = \frac{SP^2}{SQ} = \frac{SN^2 + 4AS(AS + SN)}{SQ},$$

$$\text{therefore } (SN - 2AS)^2 = 0.$$

$$\text{Hence } SN = 2AS \text{ and } AN = 3AS.$$

4. A ray of light passes from a point Q in one medium to a point S in another; its directions QR, SR , meeting the plane surface which separates them in R ; prove that $QR + \mu SR$ is least when the directions follow the law of refraction.

Draw the perpendiculars QM, SN (fig. 10); and let $QM = a$, $SN = b$, $MN = c$, also $RQM = \phi$ and $RSN = \phi'$;

then $QR + \mu SR$ is to be a minimum,

$$\text{or } \frac{a}{\cos \phi} + \frac{\mu b}{\cos \phi'} \text{ is to be a minimum.}$$

$$\text{Now } a \tan \phi + b \tan \phi' = c,$$

$$\text{therefore } \frac{a \sin \phi d\phi}{\cos^2 \phi} + \frac{\mu b \sin \phi' d\phi'}{\cos^2 \phi'} = 0 \dots \dots \dots (1),$$

$$\text{and } \frac{ad\phi}{\cos^2 \phi} + \frac{bd\phi'}{\cos^2 \phi'} = 0 \dots \dots \dots (2),$$

Using an indeterminate multiplier λ , and equating to zero the coefficients of the differentials, we have

$$\frac{a \sin \phi}{\cos^2 \phi} + \frac{a\lambda}{\cos^2 \phi} = 0,$$

$$\frac{\mu b \sin \phi'}{\cos^2 \phi'} + \frac{b\lambda}{\cos^2 \phi'} = 0 :$$

$$\text{therefore } \sin \phi = \mu \sin \phi';$$

that is, when $QR + \mu SR$ is a minimum, the directions of QR, SR follow the law of refraction.

5. If Q, q be two points in the radius of a spherical refracting surface whose centre is E , such that $EQ : Eq :: \text{sine of incidence} : \text{sine of refraction}$, determine geometrically the position of P , so that a ray proceeding from Q and incident upon the surface at P , may after refraction proceed from q .

Let P (fig. 11) be the required point, then

$$\frac{\sin QPE}{\sin qPE} = \mu,$$

$$= \frac{EQ}{Eq};$$

therefore $\frac{\sin QPE}{\sin PEQ} \cdot \frac{\sin PEq}{\sin qPE} = \frac{EQ}{Eq},$

$$\frac{EQ}{PQ} \cdot \frac{Pq}{Eq} = \frac{EQ}{Eq};$$

therefore $PQ = Pq,$

or P is determined by the intersection of the perpendicular to the radius, at the bisection of Qq , with the refracting surface.

6. A luminous point is fixed before a plane refracting surface which revolves about an axis in its own plane, find the locus of the geometrical focus.

Let Q (fig. 12) be the luminous point, AB the plane revolving about an axis at A perpendicular to the plane of the paper. Draw QB perpendicular on AB , and let q be the geometrical focus after direct refraction, then

$$Bq = \mu BQ.$$

Now if $QA = a$, $Qq = \rho$, and $AQB = \theta$,

$$\rho = (\mu - 1) a \cos \theta,$$

which represents a circle with the origin Q in the circumference, and diameter $= (\mu - 1) a$.

Ex. (A) is a fixed point. A ray PRA from P is reflected by a given plane mirror, so as to pass through A while $AR + PR$ is constant. Shew that the locus of P is a circle.

A luminous point is fixed before a plane reflecting surface which revolves about an axis in its own plane; shew that the locus of the geometrical focus is a circle.

CHAPTER III.

OBLIQUE REFLECTION AND REFRACTION.

1. A small pencil of parallel rays is obliquely reflected or refracted at a spherical surface. Shew that for a given obliquity the foci approach one another as the radius of the surface is made less.

In the case of reflection, we have (by Griffin's *Optics*, Art. 48)

$$v_1 = \frac{r}{2} \cos \phi,$$

$$v_2 = \frac{r}{2} \sec \phi,$$

therefore
$$v_1 - v_2 = \frac{r}{2} (\cos \phi - \sec \phi),$$

$$\propto r,$$

since the obliquity is constant.

For the case of refraction, we have (Griff. 57),

$$\frac{\mu \cos^2 \phi'}{v_1} = \frac{\mu \cos \phi' - \cos \phi}{r},$$

$$\frac{\mu}{v_2} = \frac{\mu \cos \phi' - \cos \phi}{r};$$

therefore
$$v_1 - v_2 = \frac{\mu r}{\mu \cos \phi' - \cos \phi} (\cos^2 \phi' - 1),$$

$$\propto r.$$

2. Find the locus of a luminous point, so that when a pencil is incident upon a small plane refracting surface, the distance between the primary and secondary foci may be constant.

If we take the centre of the small surface for origin of polar coordinates, and the normal at this point for prime radius, we have

$$v_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} u,$$

$$v_2 = \mu u,$$

$$v_1 - v_2 = \mu u \left(\frac{\cos^2 \phi'}{\cos^2 \phi} - 1 \right) = a \text{ suppose ;}$$

therefore
$$a = \mu u \frac{1 - \frac{\sin^2 \phi}{\mu^2} - 1 + \sin^2 \phi}{\cos^2 \phi},$$

$$= \frac{\mu^2 - 1}{\mu} u \tan^2 \phi ;$$

therefore
$$u = \frac{\mu}{\mu^2 - 1} a \cot^2 \phi,$$

the equation to the curve by which the required surface is generated.

3. When a pencil is incident at a very small angle on a spherical refractor, the primary focal line is nearer to or farther from the point of incidence than the secondary, according as (u) is $>$ or $<$ $(\mu + 1) r$ nearly.

By Griffin's *Optics* (Art. 51),

$$\frac{1}{v_1} = \frac{1}{\mu u} \frac{\cos^2 \phi}{\cos^2 \phi'} + \frac{1}{\mu r} \frac{\mu \cos \phi' - \cos \phi}{\cos^2 \phi'},$$

$$\frac{1}{v_2} = \frac{1}{\mu u} + \frac{1}{\mu r} (\mu \cos \phi' - \cos \phi).$$

Now
$$\begin{aligned} \frac{\cos^2 \phi}{\cos^2 \phi'} &= \left(1 - \frac{\phi^2}{2} \right)^2 \left(1 - \frac{\phi^2}{2\mu^2} \right)^{-2}, \\ &= (1 - \phi^2) \left(1 + \frac{\phi^2}{\mu^2} \right), \\ &= 1 - \frac{\mu^2 - 1}{\mu^2} \phi^2 \text{ nearly.} \end{aligned}$$

$$\begin{aligned}\text{And} \quad \mu \cos \phi' - \cos \phi &= \mu \left(1 - \frac{\phi^2}{2\mu^2} \right) - \left(1 - \frac{\phi^2}{2} \right), \\ &= (\mu - 1) \left(1 + \frac{\phi^2}{2\mu} \right);\end{aligned}$$

$$\begin{aligned}\therefore \frac{1}{v_1} &= \frac{1}{\mu u} \left(1 - \frac{\mu^2 - 1}{\mu^2} \phi^2 \right) + \frac{1}{\mu r} (\mu - 1) \left(1 + \frac{\phi^2}{2\mu} \right) \left(1 + \frac{\phi^2}{\mu^2} \right), \\ &= \frac{1}{\mu u} \left(1 - \frac{\mu^2 - 1}{\mu^2} \phi^2 \right) + \frac{1}{\mu r} (\mu - 1) \left(1 + \frac{\phi^2}{2\mu} \right) \\ &\quad + \frac{1}{\mu^3 r} (\mu - 1) \left(1 + \frac{\phi^2}{2\mu} \right) \phi^2, \\ &= \frac{1}{\mu u} + \frac{1}{\mu r} (\mu - 1) \left(1 + \frac{\phi^2}{2\mu} \right) - \frac{\mu^2 - 1}{\mu^3 u} \phi^2 + \frac{\mu - 1}{\mu^3 r} \phi^2 \text{ nearly.}\end{aligned}$$

$$\text{Now} \quad \frac{1}{v_2} = \frac{1}{\mu u} + \frac{1}{\mu r} (\mu - 1) \left(1 + \frac{\phi^2}{2\mu} \right),$$

$$\text{therefore} \quad \frac{1}{v_1} = \frac{1}{v_2} - \frac{\mu - 1}{\mu^3} \phi^2 \left(\frac{\mu + 1}{u} - \frac{1}{r} \right).$$

And if u be $> (\mu + 1) r$, $\frac{1}{v_1} - \frac{1}{v_2}$ is positive,

if u be $< (\mu + 1) r$, $\frac{1}{v_1} - \frac{1}{v_2}$ is negative;

in the first case v_1 is less than v_2 , and in the second v_1 is greater than v_2 .

4. Rays diverging from a point are incident on a plane reflecting curve. Find the nature of the curve when the distance of the primary focus from the point of incidence (or v_1) = nu .

Let r be the radius of curvature at the point of incidence, ϕ the angle; then v_1 is given by the equation

$$\frac{1}{v_1} = \frac{2}{r \cos \phi} - \frac{1}{u} \quad (\text{Griffin, 48}).$$

$$\text{therefore} \quad v_1 = \frac{u \cdot r \cos \phi}{2u - r \cos \phi} = nu,$$

therefore
$$r = \frac{2n}{n+1} u \sec \phi.$$

But if p be the perpendicular from the luminous point on the tangent at the point of incidence,

$$\sec \phi = \frac{u}{p},$$

therefore
$$r = \frac{2n}{n+1} \cdot \frac{u^2}{p} = u \frac{du}{dp} \text{ (Differential Calculus),}$$

therefore
$$\frac{2n}{n+1} \cdot \frac{dp}{p} = \frac{du}{u};$$

and integrating

$$\frac{2n}{n+1} \log p = \log \frac{u}{C};$$

therefore
$$u = Cp^{\frac{2n}{n+1}},$$

when $n = 1$, this equation becomes

$$u = Cp,$$

which represents a logarithmic spiral when referred to p and r .

CHAPTER IV.

COMBINED REFLECTIONS AND REFRACTIONS.

1. If a ray of light after being reflected any number of times, in one plane, at any number of plane surfaces, return to its former course; prove that the same will be true of any ray, parallel to the former, which is reflected at the same surfaces in the same order, provided the number of reflections be even.

Let $ABCD \dots$ (fig. 13) be the course of a ray reflected at any number of faces; $\alpha, \beta, \gamma, \delta \dots$ the angles of incidence at $A, B, C, D \dots$ respectively; $abcd \dots$ the course of a parallel ray.

Draw the perpendiculars $mm', nn' \dots$ and let $Aa = a$, then

$$mm' = Aa \cos \alpha = a \cos \alpha,$$

$$bB = \frac{mm'}{\cos \beta} = \frac{a \cos \alpha}{\cos \beta},$$

$$nn' = bB \cos \beta = a \cos \alpha = mm'.$$

Hence the distance between the rays is constant, and $= a \cos \alpha$.

Now if E and e be two corresponding points, and ϵ be the angle of incidence,

$$Ee = \frac{a \cos \alpha}{\cos \epsilon}.$$

Since the ray $ABCD \dots$ returns to A , E and A are coincident, and $\epsilon = \alpha$; therefore

$$Ee = a = Aa.$$

And if the number of reflections be even, Ee is measured in the same direction as Aa , and therefore e coincides with a .

2. If the angle of a hollow cone polished internally be any submultiple of 180° , a cylindrical pencil of rays, incident parallel to the axis, will after a certain number of reflections be a cylindrical pencil parallel to the axis, and of the same diameter as the incident pencil.

This will be the case if any ray QA (fig. 14) after $2m$ reflections falls perpendicular to AV , or after $2m + 1$ falls perpendicular to BV (AVB being a section of the cone through the axis); for in both cases the ray will trace back its course.

Let $AVB = 2\theta$ and RS a ray perpendicular to AV after $2m$ reflections; draw RT parallel to the axis VC , then the deviation TRS

$$= 2m.2\theta, \quad (\text{Griffin, 58})$$

$$= \frac{\pi}{2} - \theta;$$

therefore
$$2\theta = \frac{\pi}{2.2m + 1}.$$

But if the ray falls in the direction RR' , perpendicular to BV , after $2m + 1$ reflections; then the deviation TRR'

$$= TRS + 2\theta,$$

$$= 2m.2\theta + 2\theta = 2(2m + 1)\theta.$$

But
$$TRR' \text{ also } = \frac{\pi}{2} - \theta,$$

therefore
$$2\theta = \frac{\pi}{2(2m + 1) + 1};$$

both of which expressions are included in

$$2\theta = \frac{\pi}{2n + 1}.$$

3. If a transparent plate lie between two media of unequal densities, each less than that of the plate; and if a ray, passing from the denser of the external media into the plate, fall at the critical angle on the third medium; the angle of incidence at the first surface of the plate will be the critical angle from the first on the third medium.

Let μ_1, μ_2 be the indices of refraction from A to B and B to C respectively, ϕ the angle of incidence.

The critical angle from B to C

$$= \sin^{-1} \mu_2,$$

and $\sin \phi = \mu_1 \sin (\sin^{-1} \mu_2) = \mu_1 \mu_2,$

therefore $\phi = \sin^{-1} \mu_1 \mu_2.$

Now index of refraction from A to C

$$= \mu_1 \mu_2,$$

and critical angle from A to C

$$= \sin^{-1} \mu_1 \mu_2 = \phi.$$

4. Find the geometrical focus of a pencil of rays refracted through a hollow glass sphere whose external and internal radii are r, r' , respectively.

Let p, q_1, q_2, q_3, q_4 be the distances from the centre of the origin of light, and the geometrical foci after the first, second, third, and fourth refractions respectively; then, by Griffin (33),

$$\frac{1}{q_1} - \frac{\mu}{p} = - \frac{\mu - 1}{r} \dots \dots \dots (1).$$

Now supposing the course of the pencil reversed,

$$\frac{1}{q_1} - \frac{\mu}{q_2} = - \frac{\mu - 1}{r'} \dots \dots \dots (2).$$

Again,

$$\frac{1}{q_2} - \frac{\mu}{q_3} = \frac{\mu - 1}{r'} \dots \dots \dots (3),$$

$$\frac{1}{q_3} - \frac{\mu}{q_4} = \frac{\mu - 1}{r} \dots \dots \dots (4):$$

changing the signs of (2) and (4) and adding,

$$\frac{\mu}{q_4} - \frac{\mu}{p} = 2(\mu - 1) \left(\frac{1}{r'} - \frac{1}{r} \right),$$

which determines the geometrical focus at emergence.

5. A pencil of parallel rays is incident directly upon a sphere of glass and emerges after two internal reflections. Find its geometrical focus at emergence. ($\mu = 1.5$.)

Let q_1, q_2, q_3, q_4 , be the distances from the centre of the geometrical foci after the first refraction, the first reflection, the second reflection, and the second refraction, respectively; then

$$\text{(Griff. 33)} \quad \frac{1}{q_1} = -\frac{\mu - 1}{r} \dots\dots\dots(1),$$

$$\text{(Griff. 31)} \quad \frac{1}{q_2} + \frac{1}{q_1} = -\frac{2}{r} \dots\dots\dots(2),$$

$$\frac{1}{q_3} + \frac{1}{q_2} = \frac{2}{r} \dots\dots\dots(3),$$

$$\frac{1}{q_3} - \frac{\mu}{q_4} = \frac{\mu - 1}{r} \dots\dots\dots(4):$$

changing the signs of (1) and (3) and adding,

$$\begin{aligned} -\frac{\mu}{q_4} &= 2 \left\{ \frac{\mu - 1}{r} - \frac{2}{r} \right\}, \\ &= \frac{2(\mu - 3)}{r}. \end{aligned}$$

And since $\mu = 1.5$, we have

$$q_4 = \frac{r}{2}.$$

6. A pencil of parallel rays incident upon a sphere emerges after n internal reflections; prove that the primary focus of the refracted rays divides the distance between the point of incidence of the axis and the point where it is first reflected, in the ratio $n + 1 : n - 1$.

Let QA (fig. 15) be the axis of the pencil incident at A and refracted in direction AB ; then if ϕ, ϕ' be the angles of incidence and refraction, and r the radius of the sphere,

$$\frac{\mu \cos^2 \phi'}{v_1} = \frac{\mu \cos \phi' - \cos \phi}{r} \quad \text{(Griff. 51);}$$

therefore

$$\begin{aligned}
 v_1 &= \frac{\mu \cos^2 \phi' r}{\mu \cos \phi' - \cos \phi}, \\
 &= \frac{\mu \cos \phi' \frac{AB}{2}}{\mu \cos \phi' - \cos \phi}, \\
 &= \frac{\frac{AB}{2}}{1 - \frac{\cos \phi}{\mu \cos \phi'}}.
 \end{aligned}$$

Now deviation = $2(\phi - \phi') + n(\pi - 2\phi') = D$.

And since the pencil consists at emergence of parallel rays,

$$\frac{dD}{d\phi} = 0,$$

or

$$0 = 1 - (n + 1) \frac{d\phi'}{d\phi}:$$

differentiating $\sin \phi = \mu \sin \phi'$, we have

$$0 = \cos \phi - \mu \cos \phi' \frac{d\phi'}{d\phi},$$

therefore

$$0 = \mu \cos \phi' - (n + 1) \cos \phi,$$

therefore

$$\frac{1}{n + 1} = \frac{\cos \phi}{\mu \cos \phi'};$$

therefore

$$v_1 = \frac{AB}{2n} (n + 1).$$

And

$$\begin{aligned}
 AB - v_1 &= \frac{AB}{2n} \{2n - (n + 1)\}, \\
 &= AB \frac{n - 1}{2n};
 \end{aligned}$$

therefore

$$\frac{v_1}{AB - v_1} = \frac{n + 1}{n - 1}.$$

Ex. 1. A luminous point is placed between two plane mirrors inclined to each other; shew that there is a series of images lying in a circle, and find the number of them when the inclination of the mirrors is 30° .

For such problems see Goodwin's *Elementary Course of Mathematics*.

Ex. 2. A small pencil of parallel rays enters a reflecting sphere, and after n internal reflections emerges through the same aperture; find the position of the primary focal line after the first reflection.

$$\text{Ans. } v_1 = \frac{r}{2} \cos \frac{\pi}{2} \frac{n-1}{n+1}.$$

CHAPTER V.

PRISMS.

1. ACB (fig. 16) is a section of an isosceles prism, whose refracting angle $ACB = 2a$, made by a plane perpendicular to its edge. The refractive index of the prism $= \sqrt{2}$, and a ray of light in the plane ACB , which is incident on CB at P , after refraction falls on CA at the critical angle, and being reflected emerges through the middle point of AB . Prove that

$$CP = AC \cos a \frac{\cos a - \sin a}{\cos 2a + \sin 2a}.$$

The critical angle $= \sin^{-1} \frac{1}{\mu} = \sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$,

therefore $CRP = 45^\circ = ARD$,

and $PRD = 90^\circ$.

Now $\frac{CP}{CR} = \frac{\sin CRP}{\sin CPR} = \frac{\sin 45^\circ}{\sin (45^\circ + 2a)} \dots \dots \dots \text{I.}$

$$\frac{CR}{CD} = \frac{\sin CDR}{\sin CRD} = \frac{\sin (45^\circ - a)}{\sin 45^\circ} \dots \dots \dots \text{II.}$$

Multiplying I. by II.

$$\frac{CP}{CD} = \frac{\cos a - \sin a}{\cos 2a + \sin 2a},$$

therefore $CP = AC \cos a \frac{\cos a - \sin a}{\cos 2a + \sin 2a},$

2. A ray of light is incident at an angle ϕ on a side of an isosceles prism in a plane perpendicular to the edge, and, after internal reflection at the base, emerges from the opposite side;

prove that the deviation is $2\phi + \alpha$, α being the vertical angle of the prism.

Let $PQRST$ (fig. 17) be the course of the ray, D the deviation = SOV . Also let the normals at Q and S meet in the point a , and Sa produced meet PQ produced in b ; then

$$Qab = \alpha,$$

and

$$\begin{aligned} D &= SOV, \\ &= OSb + SbO, \\ &= OSb + (Qab + aQb), \\ &= \phi + (\alpha + \phi), \\ &= 2\phi + \alpha. \end{aligned}$$

For it is manifest that the angle of emergence = angle of incidence at Q , since the prism is isosceles.

3. If there be a small speck on the middle of one of the sides of an equilateral prism, and a person place his eye close to the opposite edge in the plane drawn through the speck perpendicular to the axis of the prism; shew that he will see two specks, and find the angle between them.

Let ABE (fig. 18) be a section of the prism by a plane through the speck C perpendicular to its axis; E the eye.

$$\text{Now } BEA = 60^\circ, \therefore CEA = 30^\circ,$$

and a ray CE incident at E , on the face AE , is refracted in a direction EP , such that if $PEQ = \theta$,

$$\sin(60^\circ + \theta) = \mu \sin(60^\circ):$$

similarly, if a ray CE incident at E on the face BE , be refracted by it in direction EP' , we have as before,

$$\sin(60^\circ + \theta) = \mu \sin 60^\circ,$$

and there will be two specks seen in the directions PE and $P'E$ respectively.

$$\text{Now } PEP' = 2\theta = 2 \left(\sin^{-1} \frac{\mu \sqrt{3}}{2} - 60^\circ \right).$$

4. Two equal prisms of the same refracting substance are placed with their edges in contact; shew that a ray of given species will be just incapable of transmission through the compound prism thus formed, when the mutual inclination of the faces

$$= 2 \sin^{-1} \{ \cos \alpha - \sqrt{(\mu^2 - 1)} \sin \alpha \},$$

α being the refractive angle of each prism and μ the index for the given light.

If the angle between the prisms be such that a ray PQ , (fig. 19) incident on the first at an angle $\frac{\pi}{2}$, emerges from the second at the same angle, the condition is satisfied. Let i be this angle, ϕ' the angle of refraction at the first surface, ψ , ψ' those of emergence and incidence on the second; then

$$\phi' = \sin^{-1} \frac{1}{\mu},$$

$$\psi' = \phi' - \alpha = \sin^{-1} \frac{1}{\mu} - \alpha,$$

$$\psi = \sin^{-1} (\mu \sin \psi'),$$

$$= \sin^{-1} \mu \left(\frac{1}{\mu} \cos \alpha - \frac{\sqrt{(\mu^2 - 1)}}{\mu} \sin \alpha \right),$$

$$= \sin^{-1} \{ \cos \alpha - \sqrt{(\mu^2 - 1)} \sin \alpha \};$$

the same will hold for the second prism, therefore

$$i = 2\psi = 2 \sin^{-1} \{ \cos \alpha - \sqrt{(\mu^2 - 1)} \sin \alpha \}.$$

5. If a pencil be incident perpendicularly on the first surface of a prism, prove that the distance of the focal lines of the emergent pencil is $u(\mu^2 - 1) \tan^2 i$, i being the vertical angle of the prism,

$$v_1 = \frac{\cos^2 \psi \cdot \cos^2 \phi'}{\cos^2 \psi' \cdot \cos^2 \phi} \cdot u \quad (\text{Griffin, Art. 82}),$$

$$v_2 = u;$$

the angles ϕ' , ψ' , are known from the conditions,

$$\sin \phi = \mu \sin \phi', \quad \sin \psi = \mu \sin \psi',$$

$$\phi' + \psi' = i.$$

If the pencil be incident perpendicularly,

$$\phi = 0, \quad \therefore \phi' = 0, \quad \text{and} \quad \psi' = i;$$

therefore

$$\begin{aligned} v_1 &= \frac{\cos^2 \psi}{\cos^2 \psi'} \cdot u, \\ &= \frac{1 - \mu^2 \sin^2 i}{1 - \sin^2 i} \cdot u, \end{aligned}$$

$$v_2 = u;$$

therefore

$$\begin{aligned} v_2 - v_1 &= (\mu^2 - 1) \frac{\sin^2 i}{1 - \sin^2 i} \cdot u, \\ &= (\mu^2 - 1) \tan^2 i \cdot u. \end{aligned}$$

6. A small pencil of light, diverging from a point, is incident on a prism, so that its axis passes through it in a plane perpendicular to the edge, with minimum deviation. The length of the path of the axis within the prism being c , and the refracting angle i , shew that the distance between the primary and secondary foci at emergence is

$$\frac{\mu^2 - 1}{\mu} \left(\tan \frac{i}{2} \right)^2 \cdot c.$$

Let Q (fig. 20) be the origin of a small pencil whose axis is obliquely refracted through the prism PVR , in direction $QPRS$; and let Q_1, Q_2 be the primary and secondary foci after the first refraction; q_1, q_2 those at emergence.

Then, if ϕ, ϕ' be the angles of incidence and refraction at the first surface, and of emergence and incidence at the second, we have

$$PQ_1 = \frac{\mu \cos^3 \phi'}{\cos^3 \phi} \cdot PQ, \quad (\text{Griffin, 50}),$$

$$PQ_2 = \mu PQ.$$

Now the pencil emanating in the primary plane from Q_1 , diverges in that plane after refraction from q_1 : hence if we suppose its course reversed, a pencil converging to q_1 , will converge in the primary plane to Q_1 after refraction into the

prism ; therefore

$$RQ_1 = \mu \frac{\cos^2 \phi'}{\cos^2 \phi} Rq_1,$$

$$RQ_2 = \mu Rq_2;$$

or

$$c + PQ_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} Rq_1,$$

$$c + PQ_2 = \mu Rq_2;$$

therefore

$$Rq_1 = \frac{c}{\mu} \frac{\cos^2 \phi}{\cos^2 \phi'} + PQ,$$

$$Rq_2 = \frac{c}{\mu} + PQ;$$

therefore

$$\begin{aligned} Rq_2 - Rq_1 &= \frac{c}{\mu} \left(1 - \frac{\cos^2 \phi}{\cos^2 \phi'} \right), \\ &= \frac{c}{\mu} \left(\frac{\mu^2 \sin^2 \phi - \sin^2 \phi'}{\cos^2 \phi'} \right), \\ &= \frac{\mu^2 - 1}{\mu} c \cdot \tan^2 \phi', \\ &= \frac{c}{\mu} (\mu^2 - 1) \tan^2 \frac{i}{2}. \end{aligned}$$

CHAPTER VI.

LENSES.

1. GIVEN the focal length of a lens in air, and the index of refraction of the medium of which it is composed; find the focal length of the same lens in water, the index of refraction of which is $\frac{4}{3}$.

Let f_1, f_2 be the focal lengths of the lens in air and in water respectively, μ the index of refraction in air.

Now if μ' be the index of refraction from water into glass, we have (Griffin, 67)

$$\mu' = \frac{\mu}{\frac{4}{3}} = \frac{3\mu}{4},$$

therefore
$$\frac{1}{f_2} = (\mu' - 1) \left(\frac{1}{r} - \frac{1}{s} \right).$$

But
$$\frac{1}{f_1} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right),$$

therefore
$$\frac{1}{r} - \frac{1}{s} = \frac{1}{f_1(\mu - 1)},$$

and
$$\frac{1}{f_2} = \frac{\frac{3\mu}{4} - 1}{f_1(\mu - 1)}.$$

2. If t be the thickness of a thin lens at the axis, ℓ' its thickness parallel to the axis at a distance y from the axis; the focal length

$$= \frac{y^2}{2(\mu - 1)(\ell' - t)}.$$

Let f be the focal length of the lens; r, s the radii of the first and second surfaces respectively; then, by Griffin's *Optics* (Art. 89),

$$\frac{1}{f} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right).$$

Now let $AB = t$, (fig. 21) $mn = t'$, $Pn = y$, $Ao = r$, $Bo' = s$, then $t' - t = mn - AB$,

$$= (mA + An) - (mA + mB),$$

$$= An - Bm,$$

$$= r \left(1 - \cos \frac{y}{r} \right) - s \left(1 - \cos \frac{y}{s} \right),$$

$$= \frac{y^2}{2r} - \frac{y^2}{2s} \text{ nearly,}$$

$$= \frac{y^2}{2} \left(\frac{1}{r} - \frac{1}{s} \right),$$

$$\therefore \frac{1}{r} - \frac{1}{s} = \frac{2(t' - t)}{y^2},$$

and $f = \frac{y^2}{2(\mu - 1)(t' - t)}.$

3. A and B being two fixed points, and P a point such that $AP = \mu BP$, the locus of P is a circle. Shew from this property how to construct a lens of common glass, such that a direct pencil, incident from a determinate point, will be refracted without aberration.

Let μ be index for common glass, $AC = u$ (fig. 22) the distance of the given point from the required lens. Take a point B in AC such that $AC = \mu BC$, and divide AB in E so that $AE = \mu BE$; then let P be a point such that $AP = \mu PB$; join CP, PE .

Now since $\mu = \frac{AP}{BP} = \frac{AE}{BE},$

therefore PE bisects the angle APB :

also since

$$\mu = \frac{AP}{BP} = \frac{AC}{BC},$$

therefore PC bisects the exterior angle; and therefore CPE is a right angle, and P a point in the circle described upon CE as diameter.

Now if a direct pencil be incident from A , and AP be any ray of it, this ray will be refracted in the direction BP , for

$$\begin{aligned} \mu &= \frac{AP}{BP}, \\ &= \frac{\sin PBO}{\sin PAO}, \\ &= \frac{\sin APO}{\sin BPO}, \\ &= \frac{\text{sine angle of incidence}}{\text{sine angle of refraction}}; \end{aligned}$$

therefore BPO is the angle of refraction, and a pencil diverging from A will after refraction at the first surface diverge from B without aberration.

Again, if $CD = t$ be the thickness required, and we give to the second surface the radius $BD = u + t$ with centre B , the pencil will be refracted through the lens thus formed without aberration.

If r and s be the radii, we have

$$AB = AC - BC = (\mu - 1)u,$$

$$AE = \mu BE = AB - BE = (\mu - 1)u - BE;$$

therefore
$$BE = \frac{\mu - 1}{\mu + 1}u,$$

$$2r = BC + BE = \frac{2\mu}{\mu + 1}u,$$

therefore
$$r = \frac{\mu u}{\mu + 1},$$

and
$$s = u + t.$$

4. If d, d' be the distances of the centre of a double convex lens from the surfaces of the lens, and D the distance of the centres of the surfaces; prove that the radii of the two surfaces are

$$r = \frac{d'(D + d' + d)}{d' + d},$$

$$r' = \frac{d(D + d' + d)}{d' + d}.$$

Let A and B (fig. 23) be the centres of the two surfaces, C the centre of the lens; then $AB = D$, $CP = d$, $CR = d'$, $AR = r$, $BP = r'$.

$$\frac{CA}{CB} = \frac{r}{r'} = \frac{r - CR}{r' - CP} = \frac{r - d'}{r' - d},$$

therefore

$$r(r' - d) = r'(r - d'),$$

$$-rd = -r'd',$$

$$r' = \frac{rd}{d'};$$

also

$$(r' - d) + (r - d') = D,$$

therefore

$$r = D + d' - \frac{rd}{d'} + d,$$

$$r \left(1 + \frac{d}{d'} \right) = D + d' + d,$$

$$r = \frac{d'(D + d + d')}{d' + d},$$

and

$$r' = \frac{d(D + d + d')}{d' + d},$$

5. A combination consists of two convex lenses of focal length f , their distance $= \frac{2f}{3}$; find the origin, in order that a pencil may emanate from it in a state of parallel rays, after direct refraction.

Let A and B (fig. 24) be the two lenses, $AB = \frac{2f}{3}$; then, after refraction through A , the pencil must diverge from a point q' at a distance $Aq' = \frac{f}{3}$ from A : writing $\frac{f}{3}$ for v in the equation

$$\frac{1}{v} - \frac{1}{u} = -\frac{1}{f} \quad (\text{Griffin 87}),$$

we find

$$u = \frac{f}{4};$$

therefore the origin is before the first lens at a distance from it $= \frac{f}{4}$.

This combination constitutes Ramsden's eye-piece.

6. A combination consists of two convex lenses, the focal lengths of the first $= 3f$, that of the second $= f$, and their distance $= 2f$; find the origin in order that a pencil may emanate from it in a state of parallel rays, after direct refraction.

Let A and B (fig. 25) be the two lenses; after refraction through the first lens, the pencil must converge to a point q' at a distance $Bq' = f = Aq'$; therefore in the equation

$$\frac{1}{v} - \frac{1}{u} = -\frac{1}{f},$$

writing $(-f)$ for v and $3f$ for f , we find

$$u = -\frac{3f}{2};$$

therefore the origin is between the lenses at a distance from the first $= \frac{3f}{2}$.

This combination constitutes Huyghen's eye-piece.

7. A pencil of rays is directly refracted through a series of lenses separated by finite intervals $a_1, a_2, a_3, \dots, a_{n-1}$; the axes being coincident. Shew that if $\frac{1}{k_1}, \frac{1}{k_2}, \frac{1}{k_3}, \dots, \frac{1}{k_n}$ be the focal

lengths of the lenses, the geometrical focus will be given by the equation

$$u_n = \frac{1}{k_n} + \frac{1}{a_{n-1}} + \frac{1}{k_{n-1}} + \dots + \frac{1}{k_1} + \frac{1}{u},$$

$$\frac{1}{v_1} - \frac{1}{u} = k_1,$$

$$\frac{1}{v_2} - \frac{1}{a_1 + v_1} = k_2,$$

$$\frac{1}{v_3} - \frac{1}{a_2 + v_2} = k_3,$$

.....

$$\frac{1}{v_n} - \frac{1}{a_{n-1} + v_{n-1}} = k_n,$$

$$\therefore v_n = \frac{1}{k_n + \frac{1}{a_{n-1} + v_{n-1}}},$$

$$= \frac{1}{\frac{1}{k_n} + \frac{1}{a_{n-1}} + \frac{1}{k_{n-1}} + \frac{1}{a_{n-2}} + \frac{1}{k_{n-2}} + \dots + \frac{1}{k_1} + \frac{1}{u}}.$$

Ex. 1. The thickness of a lens being t ; r, s the radii of its first and second surfaces; u, v the distances of the origin and geometrical focus from the first and second surfaces respectively, shew that

$$\frac{1}{\frac{\mu-1}{s} + \frac{1}{v}} - \frac{1}{\frac{\mu-1}{r} + \frac{1}{u}} = \frac{t}{\mu}.$$

Ex. 2. A small pencil of rays diverges from a point in the axis of a double convex lens, the thickness of which equals one of its radii; find an equation for determining the geometrical focus.

8. If f_1, f_2 be the focal lengths of two convex lenses at a distance a , shew that the focal length of the equivalent lens is $\frac{f_1 f_2}{f_1 + f_2 - a}$.

Let A, B (fig. 26) be the two lenses, $Aq_1 = f_1$, $AB = a$, $AC = y$, $BD = y'$; then a cylindrical pencil incident at C , parallel to the axis of the lenses, will converge to q_1 after refraction through the first; and if it converges to q_2 after the second refraction, we have, supposing the course reversed,

$$\frac{1}{Bq_1} - \frac{1}{u} = -\frac{1}{f_2},$$

therefore

$$\frac{1}{u} = \frac{1}{f_2} + \frac{1}{f_1 - a},$$

therefore

$$u = \frac{f_2(f_1 - a)}{f_1 + f_2 - a} = Bq_1.$$

Now

$$\frac{y}{y'} = \frac{Ac'}{Bq'} = \frac{f_1}{f_1 - a},$$

$$\tan Dq_2B = \frac{y'}{u};$$

therefore if F be the focal length of the equivalent simple lens, we must have

$$\tan Dq_2B = \frac{y}{F} = \frac{y'}{u},$$

or

$$\begin{aligned} F = \frac{y}{y'} \quad u &= \frac{f_1}{f_1 - a} \cdot \frac{f_2(f_1 - a)}{f_1 + f_2 - a}, \\ &= \frac{f_1 f_2}{f_1 + f_2 - a}. \end{aligned}$$

9. A combination consists of two lenses of focal length f , at a distance $\frac{2}{3}f$. If an excentrical pencil, whose axis is parallel to the axis of the lenses, be incident on the first at a distance y from the axis, find the angle between its direction at emergence and the axis of the lenses. Find also the equivalent simple lens.

Let A, B , (fig. 27) be the two lenses, QP the axis of the excentrical pencil; $Aq = f$, $AB = \frac{2f}{3}$, $AP = y$.

Then $Bq = \frac{f}{3}$ and Bq' is known from

$$\frac{1}{Bq} - \frac{1}{Bq'} = -\frac{1}{f},$$

therefore

$$\frac{1}{Bq'} = \frac{4}{f},$$

$$Bq' = \frac{f}{4},$$

$$\begin{aligned} \tan Sq'B &= \frac{BS}{Bq'} = \frac{y \frac{Bq}{Aq}}{Bq'} = \frac{y \cdot \frac{f}{3}}{\frac{f}{4} \cdot f}, \\ &= \frac{4y}{3f}. \end{aligned}$$

If F be the focal length of the equivalent lens,

$$\frac{y}{F} = \tan Sq'B = \frac{4y}{3f};$$

therefore

$$F = \frac{3f}{4}.$$

10. If an excentrical pencil, whose axis is parallel to the axis of two lenses of focal lengths $3f$ and f , and distant $2f$, be incident on the first at a distance y from the axis, find the angle between its direction at emergence and the axis of the lenses. Also find the equivalent simple lens.

Let A, B (fig. 27) be the two lenses, QP the axis of the excentrical pencil, $AP = y$, $AB = 2f$, $Aq = 3f$; then $Bq = f$, and Bq' is found from

$$\frac{1}{Bq} - \frac{1}{Bq'} = -\frac{1}{f},$$

therefore

$$Bq' = \frac{f}{2}.$$

Now
$$\tan Sq'B = \frac{BS}{Bq'} = \frac{1}{Bq'} \cdot \frac{AP \cdot Bq}{Aq},$$

$$= y \frac{f}{3f \cdot \frac{f}{2}} = \frac{2y}{3f}.$$

Let now F be the focal length of the equivalent simple lens, then

$$\frac{y}{F} = \tan Sq'B = \frac{2y}{3f};$$

therefore

$$F = \frac{3f}{2}.$$

CHAPTER VII.

ABERRATION, LEAST CIRCLE OF ABERRATION, AND CIRCLE
OF LEAST CONFUSION.

1. PARALLEL rays are reflected from a concave reflector whose radius is r and breadth $2b$, shew that the diameter of the least circle of aberration is $\frac{b^3}{4r^2}$.

Let O (fig. 28) be the centre of the reflector, $AK = b$, $AH = \frac{b}{2}$ and F the principal focus.

By Griffin's *Optics* (Art. 30),

$$\begin{aligned}\frac{1}{v_1} &= \frac{2}{r} + \frac{y^2}{r^3}, \\ &= \frac{2}{r} \left(1 + \frac{y^2}{2r^2} \right); \end{aligned}$$

or

$$\frac{1}{Ar} = \frac{1}{AF} \left(1 + \frac{y^2}{2r^2} \right);$$

therefore

$$AF - Ar = Ar \frac{y^2}{2r^2},$$

$$Fr = Ar \frac{y^2}{2r^2}.$$

But

$$tm = \frac{1}{4} \frac{Fr}{Ar} y \text{ (Art. 37);}$$

therefore

$$2tm = \frac{1}{4} \frac{y^3}{r^2},$$

or, writing b for y , diameter $= \frac{1}{4} \cdot \frac{b^3}{r^2}$.

2. A pencil of parallel rays is directly refracted at a spherical surface; shew that the radius of the least circle of aberration is $\frac{y^2}{8\mu^2 r^2}$.

The same notation being adopted as that used in Griffin's *Optics*, we have

$$\frac{\mu}{v} = \frac{\mu - 1}{r} + \frac{\mu - 1}{2\mu^2} \cdot \frac{y^2}{r^3}, \quad (\text{Art. 34}),$$

also
$$tm = \frac{1}{4} \frac{Fr}{Ar} \cdot y. \quad (\text{Art. 37}).$$

Hence
$$\frac{\mu}{Ar} = \frac{\mu - 1}{r} \left(1 + \frac{y^2}{2\mu^2 r^2} \right).$$

But
$$\frac{\mu}{AF} = \frac{\mu - 1}{r};$$

therefore
$$\frac{\mu}{Ar} = \frac{\mu}{AF} \left(1 + \frac{y^2}{2\mu^2 r^2} \right),$$

$$AF - Ar = Ar \frac{y^2}{2\mu^2 r^2} = Fr,$$

therefore
$$tm = \frac{1}{4} \frac{Fr}{Ar} \cdot y = \frac{y^3}{8\mu^2 r^2}.$$

3. A pencil is refracted directly through an equi-convex lens of focal length f , the origin and geometrical focus being equi-distant from the centre; the magnitude of the aberration is $\frac{9y^2}{2f}$, the refractive index being 1.5.

By Griffin's *Optics*, Art. 92, we have aberration

$$= -\frac{\mu - 1}{\mu^2} \left\{ \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) - \left(\frac{1}{s} - \frac{1}{v} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{v} \right) \right\} \frac{v^2 y^2}{2}.$$

Now
$$\frac{1}{f} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right);$$

therefore, writing $-r$ for r and r for s , and observing that

$$\mu - 1 = .5 = \frac{1}{2},$$

we find

$$f = -r;$$

also since

$$v = -u,$$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} = -\frac{1}{r},$$

therefore

$$u = 2r;$$

and the expression for the aberration becomes

$$\begin{aligned} &= -\frac{.5}{(1.5)^3} \left\{ \left(\frac{1}{r} + \frac{1}{u} \right)^2 \left(-\frac{1}{r} - \frac{2.5}{u} \right) - \left(\frac{1}{r} + \frac{1}{u} \right)^2 \left(\frac{1}{r} + \frac{2.5}{u} \right) \right\} \frac{u^3 y^2}{2}, \\ &= \frac{50}{225} \left(\frac{1}{r} + \frac{1}{2r} \right)^2 \times \left\{ +\frac{1}{r} + \frac{2.5}{2r} + \left(\frac{1}{r} + \frac{2.5}{2r} \right) \right\} \frac{4r^3 y^2}{2}, \\ &= \frac{50}{225} \left(\frac{9}{4r^2} \right) \times \left(\frac{9}{2r} \right) \frac{4r^3 y^2}{2}, \\ &= -\frac{50 \times 81}{225} \cdot \frac{1}{2f} \cdot \frac{y^2}{2}, \\ &= -\frac{9y^2}{2f}. \end{aligned}$$

4. An equi-convex lens and a plano-convex lens are each composed of glass whose refractive index = 1.5; and when a pencil of parallel rays is incident directly on either of them, the same aberration is produced in rays equally distant from the axis. Compare the radii of the lenses.

By Griffin's *Optics* (Art. 92, Cor.) we have

$$\text{aberration} = -\frac{\mu-1}{\mu^2} \left\{ \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu+1}{u} \right) - \left(\frac{1}{s} - \frac{1}{v} \right)^2 \left(\frac{1}{s} - \frac{\mu+1}{v} \right) \right\} \frac{v^3 y^2}{2},$$

for the equi-convex lens $r = -s$; and $u = \infty$,

$$\text{therefore aberration} = -\frac{\mu-1}{\mu^2} \left\{ -\frac{1}{s^3} - \left(\frac{1}{s} - \frac{1}{v} \right)^2 \left(\frac{1}{s} - \frac{\mu+1}{v} \right) \right\} \frac{v^3 y^2}{2},$$

and for the plano-convex writing r for s , v_1 for v , and making $r = \infty$,

$$\text{aberration} = -\frac{\mu-1}{\mu^2} \left\{ -\left(\frac{1}{r} - \frac{1}{v_1} \right)^2 \left(\frac{1}{r} - \frac{\mu+1}{v_1} \right) \right\} \frac{v_1^3 y^2}{2}.$$

But they are equal by hypothesis,

$$\therefore \left\{ \frac{1}{s^3} + \left(\frac{1}{s} - \frac{1}{v} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{v} \right) \right\} v^3 = \left\{ \left(\frac{1}{r} - \frac{1}{v_1} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{v_1} \right) \right\} v_1^3;$$

and observing that $\mu = 1.5 = \frac{3}{2}$,

$$\text{and} \quad \frac{1}{v} = -(\mu - 1) \frac{2}{s} = -\frac{1}{s},$$

$$\frac{1}{v_1} = -(\mu - 1) \frac{1}{r} = -\frac{1}{2r},$$

the equation becomes

$$\left\{ \frac{1}{s^3} + \frac{4}{s^2} \left(\frac{1}{s} + \frac{5}{2s} \right) \right\} s^3 = \frac{9}{4r^2} \left(\frac{1}{r} + \frac{5}{4r} \right) 4r^3;$$

therefore

$$\frac{15}{s} = \frac{81}{4r}, \quad \frac{20}{s} = \frac{27}{r},$$

$$\frac{r}{s} = \frac{27}{20}.$$

5. When a small pencil of parallel rays is centrically and obliquely refracted through a thin lens, the magnitude of the circle of least confusion is unaffected by the form and substance of the lens, and depends only on the breadth of the incident pencil and the inclination of its axis to that of the lens.

Let A (fig. 29) be the centre of the lens, Aq_1q_2 the axis of the pencil, λ its breadth, $rpsq$ the circle of least confusion, ϕ the angle of incidence.

Then, by Griffin's *Optics* (53),

$$Ao = \frac{v_1 v_2 (1 + \cos \phi)}{v_1 \cos \phi + v_2},$$

$$pq = \lambda \frac{v_2 - v_1}{v_2 + v_1 \cos \phi},$$

$$= \lambda \cdot \frac{\frac{1}{v_1} - \frac{1}{v_2}}{\frac{1}{v_1} + \frac{1}{v_2} \cos \phi}.$$

Now by Art. 97, we have

$$\frac{1}{v_1} = \frac{\mu \cos \phi' - \cos \phi}{\cos^3 \phi} \left(\frac{1}{r} - \frac{1}{s} \right),$$

$$\frac{1}{v_2} = (\mu \cos \phi' - \cos \phi) \left(\frac{1}{r} - \frac{1}{s} \right);$$

therefore

$$pq = \lambda \frac{\frac{1}{\cos^2 \phi} - 1}{\frac{1}{\cos^2 \phi} + \cos \phi},$$

$$= \lambda \frac{1 - \cos^2 \phi}{1 + \cos^3 \phi};$$

therefore pq depends only on λ and ϕ .

6. A segment of a paraboloid of revolution is cut off by a plane perpendicular to the axis so as to form a plano-convex lens; when a pencil of rays parallel to the axis is refracted through the lens, find the aberration in terms of the thickness which is so small that its square may be neglected.

Let QP (fig. 30) be a ray incident at P , $AB = t$, PG the normal; then if $Aq = v'$, limit $Aq = v$,

$$\mu = \frac{\sin qPG}{\sin PGq} = \frac{qG}{Pp} = \frac{2a + Aq}{Aq} \text{ ultimately,}$$

$$= \frac{2a}{v} + 1;$$

therefore

$$\frac{\mu - 1}{2a} = \frac{1}{v},$$

$4a$ being the latus-rectum of the paraboloid.

Again, $\mu = \frac{qG}{Pq} = \frac{2a + v' + t}{v' + 2t}$ nearly,

$$= \frac{2a + v' + 2t - t}{v' + 2t} = 1 + (2a - t)(v' + 2t)^{-1},$$

$$\therefore \frac{\mu - 1}{2a} = \left(1 - \frac{t}{2a} \right) (v' + 2t)^{-1},$$

$$= \left(1 - \frac{t}{2a} \right) \left(\frac{1}{v'} - \frac{2t}{v'^2} \dots \right) = \frac{1}{v'} - \frac{t}{v'} \left(\frac{1}{2a} + \frac{2}{v'} \right),$$

therefore
$$\frac{1}{v} - \frac{1}{v'} = -\frac{1}{v'} \left(\frac{t}{2a} + \frac{2}{v'} \right),$$

$$v' - v = -tv \left\{ \frac{1}{2a} + \frac{2(\mu - 1)}{2a} \right\},$$

$$= -\frac{2a}{\mu - 1} \left(\frac{2\mu - 1}{2a} \right) t,$$

$$= -\frac{2\mu - 1}{\mu - 1} t.$$

Ex. 1. Prove that a pencil of rays, directly incident at a spherical surface, is refracted without aberration when

$$u = (\mu + 1) r.$$

Ex. 2. A pencil of light is incident directly on the concave surface of a spherical mirror whose radius is one foot; the distance of the origin of the pencil from the centre of the face is two feet, and the reflecting surface could be covered by a circle whose radius is three inches; find the extreme aberration.

Ex. 3. A pencil of parallel rays passes directly through a double convex lens where the radius of the second surface is six times that of the first, the index of refraction = 1.5; shew that the aberration is $\frac{15}{14} \cdot \frac{y^2}{f}$.

Ex. 4. Compare the aberrations produced under similar circumstances, in a given direct pencil, by the same lens as a plano-convex and convexo-plane lens.

CHAPTER VIII.

DISPERSION OF LIGHT.

1. A ray of light, after passing through a prism at the angle of least deviation, falls perpendicularly on the surface of another prism of a different substance: shew that the ray will be achromatized if the angle between the prisms

$$= \cos^{-1} \left(2m \sin \frac{i}{2} \cot i_1 \right);$$

where m is the ratio of the indices of refraction, and i, i_1 the refracting angles of the prisms.

Let α be the angle between the prisms, μ the index of refraction for a fixed line through the first; ϕ, ϕ' the angles of incidence and refraction at the first surface; ψ, ψ' the angles of emergence and incidence at the second surface of the axis of a pencil of light corresponding to this fixed line; therefore

$$\sin \phi = \mu \sin \phi',$$

$$\sin \psi = \mu \sin \psi',$$

$$\phi' + \psi' = i.$$

If $\mu + \delta\mu$ be the index of refraction for another fixed line, and $\phi' + \delta\phi', \psi' + \delta\psi', \psi + \delta\psi$ the values of ϕ', ψ', ψ for this line; then, since ϕ is the same for each pencil,

$$0 = \delta\mu \sin \phi' + \mu \cos \phi' \delta\phi',$$

$$\cos \psi \delta\psi = \delta\mu \sin \psi' + \mu \cos \psi' \delta\psi',$$

$$0 = \delta\phi' + \delta\psi';$$

therefore
$$\cos \psi \frac{\delta\psi}{\delta\mu} = \sin \psi' + \frac{\cos \psi' \sin \phi'}{\cos \phi'},$$

$$\cos \psi \cos \phi' \frac{\delta\psi}{\delta\mu} = \sin (\psi' + \phi') = \sin i.$$

Let the pencil be now refracted through the second prism, and let μ_1 , $\mu_1 + \delta\mu_1$ be the corresponding values of μ , $\mu + \delta\mu$; ψ_1 , $\psi_1 + \delta\psi_1$ the angles of incidence of the axes of pencils corresponding to the two fixed lines before considered, ϕ_1' the angle of incidence on the second surface of the pencil for which the index is μ_1 . Then if the two pencils be parallel at emergence from this second prism, we have, as before,

$$\cos \psi_1 \cos \phi_1' \frac{\delta\psi_1}{\delta\mu_1} = \sin i_1.$$

But

$$\delta\psi = d\psi_1,$$

therefore

$$\frac{\cos \psi \cos \phi'}{\delta\mu \sin i} = \frac{\cos \psi_1 \cos \phi_1'}{\delta\mu_1 \sin i_1} \dots \dots \dots (1).$$

Now if we consider μ , μ_1 to be the indices for mean rays, $\mu = m\mu_1$, therefore

$$\delta\mu = m\delta\mu_1.$$

Also, upon that supposition,

$$\phi = \psi = a; \quad \phi' = \psi'; \quad 2\phi' = i;$$

$$\psi_1 = \frac{\pi}{2}, \quad \phi_1' = i_1;$$

then equation (1) becomes

$$\frac{\cos a \cos \frac{i}{2}}{\sin i} = m \frac{\cos i_1}{\sin i_1},$$

$$\cos a = 2m \sin \frac{i}{2} \cot i_1,$$

therefore

$$a = \cos^{-1} \left(2m \sin \frac{i}{2} \cot i_1 \right).$$

2. An achromatic object-glass, whose focal length is seventy inches, is composed of two lenses of crown and flint glass respectively. Find the focal length of each lens; the dispersive power of crown glass being to that of flint glass as 7:10.

f_1, f_2 being the focal lengths of the lenses of crown glass and flint glass respectively, we have for condition of achromatism,

$$\frac{\delta\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1} + \frac{\delta\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2} = 0,$$

which condition can be written

$$\frac{\mu'_v - \mu'_r}{\mu' - 1} \cdot \frac{1}{f_1} + \frac{\mu''_v - \mu''_r}{\mu'' - 1} \cdot \frac{1}{f_2} = 0,$$

μ', μ'' being the refractive indices for mean rays through the two lenses.

Now
$$\frac{\mu'_v - \mu'_r}{\mu' - 1} : \frac{\mu''_v - \mu''_r}{\mu'' - 1} :: 7 : 10.$$

Hence
$$\frac{7}{f_1} + \frac{10}{f_2} = 0 \dots\dots\dots (1).$$

Also
$$\frac{1}{f_1} + \frac{1}{f_2} = -\frac{1}{70} \dots\dots\dots (2).$$

From (1) and (2), we have

$$-\frac{10}{7} \frac{1}{f_2} + \frac{1}{f_2} = -\frac{1}{70},$$

therefore
$$f_2 = 30,$$

and
$$f_1 = -21.$$

3. Shew that if an achromatic object-glass be formed for an Astronomical Telescope by two lenses in contact, one must be concave and the other convex, and that the dispersive power of the concave lens must be greater.

Let ω_1, ω_2 be the dispersive powers of the two lenses, f_1, f_2 their focal lengths, F that of the combination.

As in the preceding problem, the condition of achromatism can be written

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0,$$

therefore either f_1 or f_2 is negative.

Griffin's *Optics* (Art. 31),

$$\frac{1}{Op} = \frac{1}{OP} - \frac{2}{r}.$$

Now P being a point in the ellipse,

$$OP = \frac{a(1 - e^2)}{1 - e \cos \theta};$$

and since $e = \frac{1}{3}$, and $a = \frac{3r}{8}$,

therefore
$$OP = \frac{r}{3 - \cos \theta};$$

therefore
$$\frac{1}{op} = \frac{3 - \cos \theta}{r} - \frac{2}{r},$$

$$op = \frac{r}{1 - \cos \theta},$$

which is the polar equation to a parabola whose latus-rectum $= 2r$, and whose focus is O .

3. If a small object be placed before a spherical mirror, prove that the image and the object subtend equal angles both at the centre and at the surface of the mirror.

Let O (fig. 33) be the centre of the mirror, PO its axis; ACB the small object, acb its image.

(1). The image of A will be in AO produced, and that of B in BO produced, therefore the object and image subtend equal angles at the centre.

(2). Again, if BP be the axis of a pencil reflected in direction Pb ; then, since $BPO = APO$, Pb passes through A , and therefore the object and image subtend equal angles at the surface of the mirror.

4. A straight rod is partly immersed in water. What must be its inclination to the horizon when its apparent portions are inclined to each other at the greatest angle?

Let AB (fig. 34) be the portion of the rod under water, AB' its image. Any point P of it will be seen in the normal through P , at a point P' whose position is determined by

$$CP = \mu CP'.$$

Let $CAP = \theta, \quad CAP' = \theta',$

then
$$\frac{\tan \theta}{\tan \theta'} = \frac{CP}{CP'} = \mu,$$

and when $\theta - \theta'$ is a maximum, $\tan(\theta - \theta')$ is a maximum, since $\theta - \theta'$ is less than 90° ;

$$\tan(\theta - \theta') = \frac{\tan \theta - \tan \theta'}{1 + \tan \theta \tan \theta'} = \frac{(\mu - 1) \tan \theta}{\mu + \tan^2 \theta} :$$

equating the differential coefficient to 0, we have

$$(\mu + \tan^2 \theta) \sec^2 \theta - 2 \tan^2 \theta \sec^2 \theta = 0,$$

$$\tan^2 \theta = \mu,$$

therefore
$$\theta = \tan^{-1} \sqrt{\mu}.$$

5. A luminous semi-circular arc is below the surface of water, and is viewed by an eye, at the centre of the arc, close to the surface of the water. Find the polar equation to the image.

Let QE (fig. 35) be the axis of a pencil incident from a point Q of the luminous arc ACB ; q_1, q_2 the primary and secondary foci, $q_1 EC = \phi$.

Then if we suppose the course of the pencil reversed, a pencil converging to q_1 will have Q for primary focus, and another converging to q_2 will have Q for secondary focus;

therefore
$$EQ = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} E q_1,$$

$$EQ = \mu E q_2.$$

If ρ be the radius vector of the image, and $EQ = a$,

$$\begin{aligned}\rho &= \frac{1}{2}(Eq_1 + Eq_2), \\ &= \frac{a}{2\mu} \left(\frac{\cos^2 \phi}{\cos^2 \phi'} + 1 \right), \\ &= \frac{a}{2\mu} \left(\frac{\mu^2 \cos^2 \phi}{\mu^2 - \sin^2 \phi} + 1 \right),\end{aligned}$$

which is the required equation.

6. A straight line, perpendicular to the axis of a thin lens, is viewed by an eye applied close to the lens; shew that, in the neighbourhood of the axis, the image will be an ellipse, parabola, or hyperbola, according as the distance of the line from the lens is $>$ or $<$ than the focal length of the lens.

If f be the focal length of the lens, whether convex or concave, and u the distance of the line from it = EP (fig. 36), then for any point Q of this line,

$$EQ = EP \sec \theta = u \sec \theta,$$

θ being the angle subtended by QP at the centre of the lens.

Now if q be the image of Q and $Eq = \rho$,

$$\begin{aligned}\frac{1}{\rho} - \frac{1}{u \sec \theta} &= \frac{1}{f}, \\ \frac{1}{\rho} &= \frac{u \sec \theta + f}{f \cdot u \sec \theta}, \\ \rho &= \frac{f}{1 + \frac{f}{u} \cos \theta},\end{aligned}$$

which is the polar equation to an ellipse, parabola, or hyperbola, according as u is $>$ or $<$ f .

7. Two convex lenses have each a focal length of three inches, and are placed two inches apart. An eye which sees at a distance of 10 inches is placed close to the second lens, and views an object through the combination. Shew that the image which the eye sees is to the object as 43 : 9.

Let A, B (fig. 37) be the two lenses, PQ the object, and $pq, p'q'$ its images after the first and second refractions respectively.

Then $Bq' = 10$, and Bq is known from

$$\frac{1}{Bq'} - \frac{1}{u} = -\frac{1}{3},$$

therefore

$$\frac{1}{u} = \frac{1}{3} + \frac{1}{10},$$

$$u = \frac{30}{13} = Bq;$$

and

$$Aq = \frac{30}{13} - 2 = \frac{4}{13}.$$

Now for determining AQ , we have

$$\frac{1}{Aq} - \frac{1}{u} = -\frac{1}{3},$$

$$\frac{1}{u} = \frac{1}{3} + \frac{13}{4} = \frac{43}{12},$$

therefore

$$u = \frac{12}{43} = AQ;$$

$$\begin{aligned} \frac{\text{image}}{\text{object}} &= \frac{p'q'}{PQ} = \frac{p'q'}{pq} \cdot \frac{pq}{PQ}, \\ &= \frac{Bq'}{Bq} \cdot \frac{Aq}{AQ}, \\ &= \frac{10}{\frac{30}{13}} \cdot \frac{\frac{4}{13}}{\frac{12}{43}} = \frac{43}{9}. \end{aligned}$$

8. An object is viewed through a convex lens of 2 inches focal length. A plate of glass $\frac{1}{2}$ inch thick is placed before the lens, its nearer surface being one inch from the lens; find the position of an object that it may be seen distinctly.

Let C (fig. 38) be the lens, AB the plate, Q a point of the object in the axis of the lens, q its image after refraction through the plate; then $AB = \frac{1}{2}$ inch, $BC = 1$ inch, and Cq must = 2 inches; therefore

$$Bq = 1 \text{ inch,}$$

and AQ is known from the equation

$$Bq = u + \frac{t}{\mu}, \text{ (Griffin, Art. 68):}$$

and since $t = \frac{1}{2}$, and $\mu = \frac{3}{2}$, therefore

$$u = \frac{2}{3} \text{ inch.}$$

9. The focal length of a double convex lens is 8 inches ; a small object is placed 4 inches, and the eye 16 inches from the lens: compare the visual angles of the object and image.

Let A (fig. 39) be the lens, E the eye, PQ the object, pq its image, then $AE = 16$ inches, $AQ = 4$, and

$$\frac{1}{Aq} - \frac{1}{AQ} = -\frac{1}{8};$$

therefore

$$Aq = 8 \text{ inches,}$$

and

$$Eq = 24.$$

Now
$$\frac{\text{visual angle of object}}{\text{visual angle of image}} = \frac{\tan PEQ}{\tan pEq},$$

$$= \frac{Eq}{EQ} \cdot \frac{PQ}{pq} = \frac{Eq}{EQ} \cdot \frac{PQ}{2PQ} = \frac{3}{5}.$$

10. A cylindrical vessel, whose breadth is b , is partly filled with water ($\mu = \frac{4}{3}$), and the bottom of the vessel appears to an eye in the axis, at a distance a from the bottom, to be twice its real breadth. Find the depth of the water in the vessel.

Let E (fig. 40) be the eye in the axis EO , BC the surface of the water, q the image of A , $AB = h$.

Then

$$AB = \mu Bq,$$

therefore

$$Bq = \frac{h}{\mu},$$

$$Aq = h - \frac{h}{\mu} = h \frac{\mu - 1}{\mu}.$$

Now since the bottom of the vessel appears to have twice its real breadth, angle $qEC = 2AEC$,

therefore $qEA = qAE$, and $qE = Aq$,

therefore $Aq^2 = qE^2 = qD^2 + ED^2$,

$$\frac{h^2(\mu - 1)^2}{\mu^2} = b^2 + \left(a - h \frac{\mu - 1}{\mu}\right)^2;$$

$$0 = b^2 + a^2 - 2ah \frac{\mu - 1}{\mu},$$

$$h = \frac{b^2 + a^2}{2a} \cdot \frac{\mu}{\mu - 1},$$

$$= \frac{2(b^2 + a^2)}{a}, \text{ since } \mu = \frac{4}{3}.$$

Ex. 1. A parabola whose latus-rectum $= 2l$ is placed before a concave spherical reflecting surface, the focus and axis of the parabola being coincident respectively with the centre and axis of the surface. The parabola is concave to the surface, and an image of it is formed by direct pencils. Shew that if f be the distance of the principal focus from the surface (f being $> l$), the image will be a confocal hyperbola, whose eccentricity $= \frac{f}{f - l}$.

Ex. 2. A plane looking-glass, of thickness t , has an object placed before it at a distance u from the first surface. Find the distance between it and its image. *Ans.* $2\left(u + \frac{t}{\mu}\right)$.

Ex. 3. If the thickness of a plate of glass be one inch, and $\mu = \frac{3}{2}$, shew that the apparent distance of objects seen through the glass will be less than their real distance by $\frac{1}{3}$ of an inch.

Ex. 4. At what distance from the eye must a concave lens be placed, that the apparent linear magnitude of a small distant object may be diminished one half? *Ans.* $u = f$.

Ex. 5. A small object is placed at right angles to the axis of a concave reflector, half-way between the surface and the principal focus. Compare the linear magnitudes of the image and the object. *Ans.* 16 : 9.

Ex. 6. A cylinder is placed before a convex reflector, their axes coinciding; find the surface which is the image of the cylinder.

CHAPTER X.

TELESCOPES.

Section 1.

1. A PLANE mirror is placed before an Astronomical Telescope, perpendicular to the axis at a given distance from the object-glass; find the distance of the lenses of the telescope when the image of the object-glass is distinctly seen.

Let ACB (fig. 41), acb be the object-glass and eye-glass, f_1, f_2 their focal lengths, a the distance of the mirror from C , EDF the image of ACB after reflection; ed the image of ED after refraction; then $CD = 2a$, and Cd is known from

$$-\frac{1}{Cd} - \frac{1}{CD} = -\frac{1}{f_1},$$

therefore

$$Cd = \frac{2af_1}{2a - f_1}.$$

Also

$$cd = f_2,$$

therefore

$$Cc = f_2 + \frac{2af_1}{2a - f_1} = \frac{2a(f_1 + f_2) - f_1f_2}{2a - f_1}.$$

2. In a Gregorian telescope, the focal lengths of the mirrors are 20 inches and 1 inch; the focal length of the eye-glass = 1 inch, the foci of the mirrors being $\frac{1}{100}$ inch asunder: find the place of the eye-glass.

Let C (fig. 42) be the large mirror, E the small mirror, a the eye-glass; $pq, p'q'$ the images after the first and the second reflections respectively, $Cq = 20$, $Eq = EF + Fq = 1 + \frac{1}{100}$; F being the principal focus of the small mirror:

Then
$$\frac{1}{Eq'} + \frac{1}{Eq} = \frac{2}{r} = 1,$$

therefore
$$\frac{1}{Eq'} = 1 - \frac{100}{101},$$

$$Eq' = 101.$$

But $aq' = 1, \therefore Ea = 102,$

and $EC = Cq + Eq,$

$$= 20 + \frac{100}{100} = \frac{2100}{100},$$

therefore
$$Ca = 102 - \frac{2100}{100},$$

$$= 80\frac{99}{100} \text{ inches.}$$

3. In a Galileo's telescope the focal lengths are 6 inches and 1 inch respectively; find the distance between them, when an object 20 feet distant is viewed.

Let ACB (fig. 43) be the object-glass, acb the eye-glass, pq the image of PQ , and $CQ = 20$ feet; then

$$-\frac{1}{Cq} - \frac{1}{CQ} = -\frac{1}{6},$$

$$\frac{1}{Cq} = \frac{1}{6} - \frac{1}{240} = \frac{39}{240};$$

therefore $Cq = \frac{240}{39},$

$$Cc = Cq - 1 = \frac{201}{39} = 5\frac{2}{13} \text{ inches.}$$

4. A telescope is composed of three lenses, the focal lengths of which are $36f$, $2f$, and f ; the distance between the second and third is $\frac{3f}{2}$; find the distance between the first and second when the rays emerge parallel.

Let A, B, C (fig. 44) be the three lenses, and $Aq = 36f$, $Cq' = f$, $BC = \frac{3}{2}f$; then $Bq' = \frac{f}{2}$, and if we suppose the lens B convex,

$$\frac{1}{Bq} - \frac{1}{Bq'} = -\frac{1}{2f},$$

$$\frac{1}{Bq} = \frac{2}{f} - \frac{1}{2f} = \frac{3}{2f},$$

therefore

$$\begin{aligned} Bq &= \frac{3}{2}f, \\ AB &= 86f - \frac{3}{2}f, \\ &= (85\frac{1}{2})f. \end{aligned}$$

5. The axis of Gregory's telescope is turned towards the centre of the Sun's disc, the apparent diameter of which is β ; prove that the diameter of the image, supposed circular, formed at the centre of the face of the large mirror, is $\frac{df\beta}{d-f}$, where d is the distance between the mirrors, and f the focal length of the large mirror.

Let the Sun's centre be in the axis CE (fig. 45), and a ray from the upper limb be reflected at C in directions Cr , rY , and have its image at P ; then PC will be the radius of the bright image.

Now
$$\frac{PC}{Er} = \frac{CY}{EY};$$

also if pq be the radius of the image formed by the first reflection,

$$\frac{Er}{pq} = \frac{CE}{CY},$$

therefore
$$\frac{CP}{pq} = \frac{CE}{EY} = \frac{d}{d-f},$$

therefore
$$\begin{aligned} CP &= \frac{d}{d-f} pq = \frac{d}{d-f} Cq \times \text{angle } pCq, \\ &= \frac{d}{d-f} f \times \frac{\beta}{2}, \end{aligned}$$

therefore diameter of image $= \frac{d}{d-f} \beta f.$

6. The axis of the object-glass of an Astronomical telescope is inclined to the axis of the telescope at a small angle α . Shew that in order to adjust the telescope for viewing distant objects, the eye-glass must be pushed in through a space $fa^2 \frac{\mu + 1}{2\mu}$,

where f is the focal length of the object-glass, and μ its refractive index.

Let PA be the axis of a pencil of parallel rays incident at A and excentrically refracted; then, since the angle of incidence = a , we have, observing that $\phi = a$, $\phi' = \frac{a}{\mu}$ nearly,

$$\frac{1}{v_1} = \frac{\mu \left(1 - \frac{a^2}{2\mu^2}\right) - \left(1 - \frac{a^2}{2}\right)}{\left(1 - \frac{a^2}{2}\right)} \left(\frac{1}{r} - \frac{1}{s}\right),$$

(Griffin's *Optics*, Art. 97),

$$\begin{aligned} &= \left\{ \mu \left(1 - \frac{a^2}{2\mu^2}\right) - \left(1 - \frac{a^2}{2}\right) \right\} (1 + a^2) \left(\frac{1}{r} - \frac{1}{s}\right), \\ &= (\mu - 1) \left\{ \left(1 + \frac{a^2}{2\mu}\right) (1 + a^2) \right\} \left(\frac{1}{r} - \frac{1}{s}\right), \\ &= (\mu - 1) \left\{ 1 + a^2 \left(1 + \frac{1}{2\mu}\right) \right\} \left(\frac{1}{r} - \frac{1}{s}\right), \\ &= \frac{1}{f} + \frac{a^2(2\mu + 1)}{2\mu} \frac{1}{f}. \end{aligned}$$

Similarly

$$\frac{1}{v_2} = \frac{1}{f} + \frac{a^2}{2\mu} \cdot \frac{1}{f},$$

therefore

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{v_1} + \frac{1}{v_2}\right) &= \frac{1}{f} + \frac{a^2}{2\mu} \cdot \frac{\mu + 1}{f}, \\ &= \frac{1}{f_1}; \end{aligned}$$

if we suppose that f_1 is the distance of the image from the lens,

$$\text{then} \quad f - f_1 = fa^2 \frac{\mu + 1}{2\mu},$$

and therefore the eye-glass must be pushed in through a space $fa^2 \frac{\mu + 1}{2\mu}$.

SECT. 2.—*Magnifying Power of Telescopes.*

1. The magnifying power of an Astronomical telescope is m , find what it becomes when the instrument is adjusted for minute observation by a person whose distance of distinct vision is n times the focal length of the eye-glass.

Let A , B (fig. 46) be the object-glass and eye-glass, F , f their focal lengths, $Bq' = nf$; then

$$\frac{1}{nf} - \frac{1}{Bq} = -\frac{1}{f},$$

$$\frac{1}{Bq} = \frac{n+1}{nf}.$$

$$\begin{aligned} \text{Magnifying power} &= \frac{p'Bq'}{pAq}, \\ &= \frac{pBq}{pAq} = \frac{Aq}{Bq}, \\ &= \frac{F(n+1)}{nf}, \\ &= \frac{m(n+1)}{n}, \end{aligned}$$

since

$$m = \frac{F}{f}.$$

2. Find the magnifying power of Newton's telescope with Ramsden's eye-piece.

Let F and f be the focal lengths of the large mirror and of the lenses of the eye-piece respectively, pq , $p'q'$, $p''q''$ (fig. 47) the images after reflection at the large mirror, reflection at the plane mirror and refraction through the first lens; then

$$p'q' = pq, \quad bq'' = f, \quad aq' = \frac{f}{4}, \quad aq'' = \frac{f}{3}, \quad \text{and } Cq = F.$$

$$\begin{aligned}
 \text{The magnifying power} &= \frac{p''bq''}{pOq}, \\
 &= \frac{p''bq''}{pCq}, \\
 &= \frac{p''bq''}{p'aq''} \cdot \frac{p'aq'}{pCq}, \\
 &= \frac{aq''}{bq''} \cdot \frac{Cq}{aq'}, \\
 &= \frac{\frac{f}{3}}{\frac{f}{4}} \cdot \frac{F}{f} = \frac{4F}{3f}.
 \end{aligned}$$

3. Find the magnifying power of Gregory's telescope with Huyghen's eye-piece; and prove that it is equal to the ratio of the diameter of the large mirror to that of its bright image formed by the small mirror and eye-piece.

Let PC (fig. 48) be the imaginary ray of the pencil from P , which would be reflected at C ; and let it be reflected at r , by the small mirror, in a direction whose intersection with the axis of the mirrors is y , y being nearly coincident with the principal focus of either mirror.

$$\begin{aligned}
 \text{Then magnifying power} &= \frac{p''aq''}{pOq} = \frac{p'Cq'}{ECr}, \\
 &= \frac{p'Cq'}{p'yq'} \cdot \frac{Eyr}{ECr} = \frac{yq'}{Cq'} \cdot \frac{EC}{Ey}.
 \end{aligned}$$

Now if F_0 , f_m , $3f$, and f be the focal lengths of the large mirror, the small mirror, the field-glass, and eye-lens respectively, we have

$$yE = f_m, \quad Cq' = \frac{3f}{2}, \quad Cq'' = f, \quad EC = F_0 + f_m \text{ nearly,}$$

$$\text{therefore magnifying power} = \frac{F_0 + \frac{3f}{2}}{\frac{3f}{2}} \cdot \frac{F_0 + f_m}{f_m}.$$

Next, EC (fig. 49) being large, $a'q$ the image of AC will be formed very nearly at the principal focus of the small mirror.

Let $p'q'$ be the virtual image of $a'q$ formed by the field-glass, $p''q''$ the bright image formed by the small mirror and eye-piece;

then

$$Cq = F_0,$$

and

$$+ \frac{1}{Cq'} + \frac{1}{Cq} = + \frac{1}{3f},$$

therefore

$$\frac{1}{Cq'} = \frac{1}{3f} - \frac{1}{F_0} = \frac{F_0 - 3f}{3f.F_0};$$

also

$$\frac{1}{aq'} - \frac{1}{aq''} = - \frac{1}{f},$$

therefore

$$\frac{aq'}{aq''} = 1 + \frac{aq'}{f}.$$

$$\begin{aligned} \text{Now } \frac{\text{diameter of large mirror}}{\text{diameter of its image}} &= \frac{AC}{p''q''} = \frac{AC}{aq'} \cdot \frac{aq'}{p''q''}, \\ &= \frac{CE}{qE} \cdot \frac{aq'}{p'q'} \cdot \frac{p'q'}{p''q''} = \frac{CE}{qE} \cdot \frac{Cq}{Cq'} \cdot \frac{aq'}{aq''}, \\ &= \frac{F_0 + f_m}{f_m} \cdot \frac{F_0}{cq'} \cdot \frac{aq'}{aq''}, \\ &= \frac{F_0 + f_m}{f_m} \cdot \frac{F_0(F_0 - 3f)}{3f.F_0} \left(1 + \frac{cq' - 2f}{f} \right), \\ &= \frac{F_0 + f_m}{f_m} \cdot \frac{F_0 - 3f}{3f} \left(1 + \frac{F_0 + 6f}{F_0 - 3f} \right), \\ &= \frac{F_0 + f_m}{f_m} \cdot \frac{2F_0 + 3f}{3f} = \frac{F_0 + f_m}{f_m} \cdot \frac{F_0 + \frac{3f}{2}}{\frac{3f}{2}}, \\ &= \text{magnifying power.} \end{aligned}$$

4. Prove that the magnifying power of an Astronomical telescope, with Ramsden's eye-piece, is the ratio of the diameter of the object-glass to that of its bright image formed by the eye-piece. Shew that the diameter of this bright image is also the diameter of a direct pencil from a very distant point, on emerging from the eye-piece.

$$\begin{aligned}
 \text{Magnifying power} &= \frac{p' a q'}{p' E q'} \cdot \frac{p E q}{p A q}, \\
 &= \frac{E q'}{a q'} \cdot \frac{A q}{E q}, \\
 &= \frac{f}{3f} \cdot \frac{F_0}{\frac{3}{4}f} = \frac{F_0}{3(a - F_0)}.
 \end{aligned}$$

6. An Astronomical telescope has an object-glass whose focal length is F_0 , and the Huyghen's eye-piece has an eye-glass of focal length f ; find the magnifying power.

Let A , E , a (fig. 53) be the object-glass, field-glass, and eye-lens respectively; pq the virtual image formed by refraction through the object-glass, $p'q'$ the real image formed after refraction through the field-glass; then

$$Aq = F_0, \quad aq' = f, \quad Ea = 2f, \quad Eq = \frac{3f}{2}.$$

$$\begin{aligned}
 \text{Magnifying power} &= \frac{p' a q'}{p' E q'} \cdot \frac{p E q}{p A q}, \\
 &= \frac{E q'}{a q'} \cdot \frac{A q}{E q}, \\
 &= \frac{f}{f} \cdot \frac{F_0}{\frac{3f}{2}} = \frac{2F_0}{3f}.
 \end{aligned}$$

Ex. 1. Having given f_0 , f_m , f_e are the focal lengths of the reflectors and eye-lens of Gregory's or Cassegrain's telescope, and z the distance between the principal foci of the small reflector and eye-lens; shew that $\frac{f_0 z}{f_m f_e}$ is an accurate measure of the magnifying power.

Ex. 2. If the focal length of the mirrors of Gregory's telescope be 12 inches and 1 inch; find the aperture of the eye-glass that the field of view may be $30'$.

Ex. 3. Prove that the magnifying power of Newton's telescope is equal to the ratio of the diameter of the large mirror to that of its image formed by the eye-glass.

Ex. 4. In an Astronomical telescope with Huyghen's eye-piece, if the focal length of the object-glass is 3 feet and the magnifying power 200, what must be the focal lengths of the lenses which form the eye-piece.

Ex. 5. In an Astronomical telescope the focal length of the object-glass = 2 feet, that of the eye-glass is one inch; the aperture of the former = 4 inches, of the latter = $\frac{1}{2}$ inch. Find the magnitude of the stop that will cure the ragged edge.

Ans. Diameter of stop = $\frac{8}{15}$.

Ex. 6. The focal length of the object-glass of a simple Astronomical telescope is 15 inches and its breadth 3 inches. Find the focal length of the eye-lens in order that in looking at a distant point, the pencil of rays may just fill the pupil of the eye; the breadth of the pupil being $\frac{1}{2}$ of an inch.

Ans. $f = 1$ inch.

Ex. 7. If the focal lengths of the lenses of an Astronomical telescope be 12 inches and 1 inch; by how much must the eye-glass be pulled out for viewing an object at a distance 40 feet from the object-glass. *Ans.* $\frac{1}{3}$ of an inch.

Ex. 8. When the same micrometer is used for different telescopes, shew that the value of a turn of the screw is inversely as the focal length of the object-glass.

CHAPTER XI.

THE RAINBOW.

1. THE angular breadths of the primary and secondary rainbows, at the eye of the observer, are respectively

$$\frac{2}{\mu} \left(\frac{4 - \mu^2}{\mu^2 - 1} \right)^{\frac{1}{2}} \delta\mu \quad \text{and} \quad \frac{2}{\mu} \left(\frac{9 - \mu^2}{\mu^2 - 1} \right)^{\frac{1}{2}} \delta\mu$$

nearly, where $\delta\mu$ is the difference of the index of refraction for the extreme rays, and μ the index for mean rays.

The deviation after n internal reflections

$$= 2(\phi - \phi') + n(\pi - 2\phi') = D, \text{ (Griffin, 221),}$$

and the rays being parallel at emergence,

$$\frac{dD}{d\phi} = 0 = \mu \cos \phi' - (n + 1) \cos \phi. \dots\dots\dots(1).$$

Let D_1 be the deviation corresponding to the extremity of the colour considered. In examining its alteration produced by an alteration of μ , ϕ' and ϕ must be regarded as functions of μ ,

therefore
$$\frac{dD_1}{d\mu} = 2 \left\{ \frac{d\phi}{d\mu} - (n + 1) \frac{d\phi'}{d\mu} \right\}.$$

But
$$\cos \phi \frac{d\phi}{d\mu} = \sin \phi' + \mu \cos \phi \frac{d\phi'}{d\mu},$$

$$\therefore \frac{dD_1}{d\mu} = \frac{2}{\cos \phi} \left\{ \sin \phi' + \mu \cos \phi \frac{d\phi'}{d\mu} - (n + 1) \cos \phi \frac{d\phi'}{d\mu} \right\},$$

$$= \frac{2 \sin \phi'}{\cos \phi} \text{ from (1),}$$

$$= \frac{2}{\mu} \tan \phi;$$

therefore
$$dD_1 = \frac{2}{\mu} \tan \phi \, d\mu,$$

or
$$\delta D_1 = \frac{2}{\mu} \tan \phi \, \delta\mu:$$

for the primary rainbow

$$\begin{aligned} \delta D_1 &= \frac{2}{\mu} \frac{\sqrt{\left(\frac{4 - \mu^2}{3}\right)}}{\sqrt{\left(1 - \frac{4 - \mu^2}{3}\right)}} \delta\mu, \\ &= \frac{2}{\mu} \left(\frac{4 - \mu^2}{\mu^2 - 1}\right)^{\frac{1}{2}} \delta\mu; \end{aligned}$$

for the secondary rainbow

$$\begin{aligned} \delta D_1 &= \frac{2}{\mu} \frac{\sqrt{\left(\frac{9 - \mu^2}{8}\right)}}{\sqrt{\left(1 - \frac{9 - \mu^2}{8}\right)}} \delta\mu, \\ &= \frac{2}{\mu} \left(\frac{9 - \mu^2}{\mu^2 - 1}\right)^{\frac{1}{2}} \delta\mu; \end{aligned}$$

and these are the angular breadths of the primary and secondary rainbows respectively, for the angular breadth is equal in either case to the change in deviation corresponding to $\delta\mu$.

2. Investigate the order of the colours in the primary rainbow; shew that if A be the sun's altitude, and

$$\sin \frac{A}{2} > \frac{1}{\mu^2} \left(\frac{4 - \mu^2}{3}\right)^{\frac{3}{2}},$$

the bow will not be visible.

For the first part of this problem, see Griffin's *Optics*, Art. 221.

If D_1 be the minimum deviation of light of a given kind, a the altitude of the corresponding colour in the primary bow; then

$$D_1 + a = \pi - \text{altitude of the sun's centre,}$$

therefore if the bow is invisible, α is zero or negative, and

$$A \leq \pi - D_1,$$

therefore
$$\frac{A}{2} \geq \sin \left(\frac{\pi}{2} - \frac{D_1}{2} \right).$$

But
$$\begin{aligned} D_1 &= 2(\phi - \phi') + (\pi - 2\phi'), \\ &= 2\phi + \pi - 4\phi', \end{aligned}$$

therefore
$$\begin{aligned} \sin \frac{A}{2} &\geq \sin(2\phi' - \phi), \\ &\geq \sin 2\phi' \cos \phi - \cos 2\phi' \sin \phi. \end{aligned}$$

Now
$$\sin \phi = \sqrt{\left(\frac{4 - \mu^2}{3} \right)},$$

$$\begin{aligned} \therefore \sin \frac{A}{2} &\geq \frac{2}{\mu} \sin \phi \cos \phi' \cos \phi - \left(1 - \frac{2 \sin^2 \phi}{\mu^2} \right) \sin \phi, \\ &\geq \frac{2}{\mu} \sqrt{\left(\frac{4 - \mu^2}{3} \right)} \times \sqrt{\left(1 - \frac{4 - \mu^2}{3\mu^2} \right)} \times \sqrt{\left(1 - \frac{4 - \mu^2}{3} \right)} \\ &\quad - \left(1 - 2 \frac{4 - \mu^2}{3\mu^2} \right) \sqrt{\left(\frac{4 - \mu^2}{3} \right)}, \\ &\geq \frac{2}{\mu} \sqrt{\left(\frac{4 - \mu^2}{3} \right)} \left\{ \sqrt{\left(\frac{4\mu^2 - 4}{3\mu^2} \right)} \cdot \sqrt{\left(\frac{4 - \mu^2}{3} \right)} - \frac{5\mu^2 - 8}{6\mu} \right\}, \\ &\geq \frac{1}{3\mu^3} \sqrt{\left(\frac{4 - \mu^2}{3} \right)} (4\mu^2 - 4 - 5\mu^2 + 8), \\ &\geq \frac{1}{\mu^3} \left(\frac{4 - \mu^2}{3} \right)^{\frac{3}{2}}. \end{aligned}$$

3. Determine the altitude of the primary rainbow, and find the breadth of the bow of any colour: having given

$$3 \cos^2 \phi = \mu^2 - 1,$$

$$3 \cos^2 \phi = \mu^2 - 1, \therefore \sin \phi = \sqrt{\left(\frac{4 - \mu^2}{3} \right)},$$

and
$$\sin \phi' = \sqrt{\left(\frac{4 - \mu^2}{3\mu^2} \right)}.$$

$$\begin{aligned}
 \text{Deviation} &= 2(\phi - \phi') + \pi - 2\phi', \\
 &= \pi + 2\phi - 4\phi', \\
 &= \pi + 2 \sin^{-1} \sqrt{\left(\frac{4 - \mu^2}{3}\right)} - 4 \sin^{-1} \sqrt{\left(\frac{4 - \mu^2}{3\mu^2}\right)}.
 \end{aligned}$$

Now if α be the Sun's altitude, A the altitude of the bow for the colour whose index of refraction is μ ,

$$A + \alpha = \pi - D,$$

$$\text{therefore } A = 4 \sin^{-1} \sqrt{\left(\frac{4 - \mu^2}{3\mu^2}\right)} - 2 \sin^{-1} \sqrt{\left(\frac{4 - \mu^2}{3}\right)} - \alpha,$$

and if μ_1, μ_2 be the indices for two consecutive colours, μ_1 being $< \mu_2$, and therefore corresponding to the colour which has the greater altitude of the two, and A_1, A_2 be the altitudes of the bow for these colours, then the breadth of the bow for the colour whose refractive index is μ_1 ,

$$\begin{aligned}
 = A_1 - A_2 &= 4 \left\{ \sin^{-1} \sqrt{\left(\frac{4 - \mu_1^2}{3\mu_1^2}\right)} - \sin^{-1} \sqrt{\left(\frac{4 - \mu_2^2}{3\mu_2^2}\right)} \right\} \\
 &\quad - 2 \left\{ \sin^{-1} \sqrt{\left(\frac{4 - \mu_1^2}{3}\right)} - \sin^{-1} \sqrt{\left(\frac{4 - \mu_2^2}{3}\right)} \right\}.
 \end{aligned}$$

CHAPTER XII.

MISCELLANEOUS PROBLEMS.

1. A ray of light is incident at a given angle upon a sphere in which the index of refraction varies inversely as the square of the distance from the centre ; determine the path described by the ray.

The ray will continue in the plane which passes through the direction before incidence and the centre of the sphere. Let this plane be the plane of the paper. Suppose the sphere divided into an infinitely large number of shells each of uniform density ; PQ , QR (fig. 54) any two consecutive ones, PQR the path of the ray.

Let $OP = r$, $OQ = r - \delta r$, μ = index of refraction in $PQ = \frac{\nu}{r^2}$;
 $\mu + \delta\mu$ that in $QR = \frac{\nu}{(r - \delta r)^2} = \frac{\nu}{r^2} \left(1 - \frac{\delta r}{r} \right)^{-2} = \frac{\nu}{r^2} \left(1 + \frac{2\delta r}{r} \right)$;
 $PQN = \phi$, $OQR = \phi - \delta\phi$; then

$$\frac{\sin \phi}{\sin (\phi - \delta\phi)} = \frac{\mu + \delta\mu}{\mu} = 1 + \frac{2\delta r}{r} ;$$

$$\text{therefore } \frac{\sin \phi}{\sin \phi - \delta\phi \cos \phi} = \frac{1}{1 - \delta\phi \cot \phi} = 1 + \delta\phi \cot \phi ;$$

$$\text{therefore } d\phi \cot \phi = \frac{2dr}{r} ,$$

$$\log \sin \phi = \log r^2 - \log c^2 ,$$

$$\sin \phi = \frac{r^2}{c^2} \dots\dots\dots (1).$$

But if a , a' be the angles of incidence and refraction, respectively,

at a distance a from 0, then

$$\sin a' = \frac{\sin a}{\mu} = \frac{a^2}{v} \sin a,$$

$$= \frac{a^2}{c^2} \text{ from (1);}$$

therefore

$$\sin \phi = \frac{r^2 \sin a}{v} \dots\dots\dots (2).$$

Let θ be the angle that OQ makes with the initial radius; then

$$\sin \phi = \frac{rd\theta}{\sqrt{\{dr^2 + (rd\theta)^2\}}} = \frac{r^2 \sin a}{v} \text{ by (2),}$$

therefore

$$(d\theta)^2 = \frac{r^2 \sin^2 a}{v^2} (dr^2 + r^2 d\theta^2),$$

$$d\theta^2 \left(1 - \frac{r^4 \sin^2 a}{v^2} \right) = \frac{r^2 \sin^2 a}{v^2} dr^2,$$

$$d\theta = \frac{\pm r dr}{\sqrt{(v^2 \operatorname{cosec}^2 a - r^4)}},$$

$$= \frac{-\frac{1}{2} d(r^2)}{\sqrt{\{v^2 \operatorname{cosec}^2 a - (r^2)^2\}}};$$

since θ increases when r diminishes, therefore

$$\theta = \frac{1}{2} \cos^{-1} \frac{r^2}{v \operatorname{cosec} a} + C.$$

Let the initial radius be such that $C = 0$, then

$$r^2 = v \operatorname{cosec} a \cdot \cos 2\theta,$$

the required equation.

2. ABC, ABD are the faces of a triangular prism of small refracting angle a , and the refractive index out of air at any point P in the prism $= \mu(1 + m\theta)$, where m is constant and θ is the angle between the planes ABC and ABP . Shew that, if D be the deviation of a ray which is incident perpendicularly on ABC and refracted through the prism, $D = (\mu - 1)a + \frac{\mu m a^2}{2}$.

We may suppose the prism of which DBC (fig. 55) is a section, by a plane perpendicular to the axis, to be made up of

a number of small prisms separated by an infinitely small space, and having a common vertex.

Let PBR be a section of one of these, with refracting angle $\delta\theta$; then, if δD be the deviation through this prism, we have, since the refractive index $= \mu(1 + m\theta)$,

$$\delta D = \{\mu(1 + m\theta) - 1\} \delta\theta;$$

therefore, proceeding to the limit,

$$dD = \{\mu - 1 + m\mu\theta\} d\theta.$$

And, since the whole deviation must be the sum of all such deviations, we have, by integrating,

$$\begin{aligned} D &= \int_0^a \{(\mu - 1) d\theta + \mu m\theta d\theta\}, \\ &= (\mu - 1) a + \frac{1}{2} \mu m a^2, \end{aligned}$$

since when $\theta = 0$, deviation also $= 0$.

3. Find the caustic when the curve is a circle and the origin of light is in the circumference.

Let A (fig. 56) be the origin, P a point in the caustic; then if $AB = u$, $BP = v_1$, $AO = r$, and ϕ be the angle of incidence,

$$\frac{1}{v_1} + \frac{1}{u} = \frac{2}{r \cos \phi} \quad (\text{Griffin, Art. 48}).$$

But

$$u = 2r \cos \phi;$$

therefore

$$\frac{1}{v_1} = \frac{3}{2r \cos \phi},$$

$$v_1 = \frac{u}{3}.$$

Produce BP to meet the circumference in C , and join OC . Trisect BO in M and N , and draw MP , ND parallel to OC ; then the triangles BMP and BOC are similar, and

$$BM = MP = MN.$$

With O as centre and radius ON describe a circle, and with centre M and radius MN describe another; then angle

$$BMP = BOC = BOA,$$

therefore angle $PMN = NOK$, and $MN = ON$,

therefore $\text{arc } NP = \text{arc } NK$.

And the locus of P is an epicycloid, generated by the rolling of a circle of radius $\frac{r}{3}$, upon an equal circle concentric with the reflecting curve.

4. Find the caustic when the reflecting curve is a cycloid, and the incident rays are parallel to the axis.

Let ABC (fig. 57) be a cycloid, QR any ray incident parallel to the axis, and KRH the generating circle. Draw the diameter ROG , join RH ; draw HP perpendicular to RG , and let $QRH = \phi$.

Now $ORH = OHR = QRH$,

therefore ROG is the direction of the reflected ray. If r be the radius of curvature at R , we have

$$\frac{1}{v_1} = \frac{2}{r \cos \phi};$$

but $r = 2RH$, therefore $v_1 = RH \cos \phi = PR$; and therefore P is a point in the caustic.

Bisect OH in S , and with centre S and radius OS describe a circle. This circle will pass through P , since OPH is a right angle. Join SP ; then angle $PSH = 2GOH$, and $SP = \frac{1}{2}OG$, therefore $\text{arc } PH = \text{arc } GH = HB$; therefore the locus of P is a cycloid CPB , and the diameter of its generating circle is half that of the given cycloid.

5. A ray of light is incident upon one of two reflectors, inclined to each other at an angle $\frac{\pi}{n}$ in a direction parallel to a line which is at right angles to their intersection, and bisects the angle between them. Supposing the intensity of a ray reflected at an angle ϕ to be to that of the incident ray as $e \cos \phi$ to 1, shew that the intensity of the ray, after it has suffered n reflections, will be to that of the incident ray as e^n to 2^{n-1} .

Let PVQ (fig. 58) be a section of the reflectors by a plane perpendicular to their intersection, VA a line bisecting the angle between them; $\phi_1, \phi_2, \phi_3, \dots \phi_n$ the angles of incidence at the 1st, 2nd, 3rd, $\dots$ n^{th} reflections; then

$$\phi_1 = \frac{\pi}{2} - PVA = \frac{\pi}{2} - \frac{\pi}{2n},$$

$$\phi_2 = \phi_1 - \frac{\pi}{n},$$

$$\phi_3 = \phi_2 - \frac{\pi}{n},$$

$$\dots$$

$$\phi_n = \phi_{n-1} - \frac{\pi}{n}.$$

By addition,

$$\begin{aligned} \phi_n &= \frac{\pi}{2} - \frac{\pi}{2n} - (n-1) \frac{\pi}{n}, \\ &= \frac{\pi}{2} - \frac{2n-1}{2n} \pi. \end{aligned}$$

Let $RS, QR, PQ \dots$ be the directions of a ray after the $n^{\text{th}}, (n-1)^{\text{th}}, (n-2)^{\text{th}} \dots$ reflections respectively; then

$$\frac{\text{intensity of } RS}{\text{intensity of } QR} = e \cos \phi_n,$$

$$\frac{\text{intensity of } QR}{\text{intensity of } PQ} = e \cos \phi_{n-1},$$

$$\dots$$

and by multiplication,

$$\begin{aligned} &\frac{\text{intensity of the ray after } n \text{ reflections}}{\text{intensity of the incident ray}} \\ &= e^n \cos \phi_1 \cos \phi_2 \dots \cos \phi_n, \\ &= e^n \sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \sin \frac{5\pi}{2n} \dots \sin \frac{2n-1}{2n} \pi, \\ &= \frac{e^n}{2^{n-1}}. \end{aligned}$$

Vide Hymers' *Theory of Equations*, p. 29, 2nd edit.

6. If the law of refraction were assumed to be $\phi = \mu\phi'$, shew that the approximate error in finding the point where a ray, incident near and parallel to the axis of a spherical refracting surface, cuts the axis after refraction, would be $\frac{1}{6\mu} \cdot \frac{\mu+1}{\mu-1} \cdot \frac{y^2}{r}$, using the ordinary notation.

Let the ray PQ (fig. 59) after refraction cut the axis AO in q , and let $AP = y$, $AO = r$, $Aq = v'$, and limit $v' = v$; then, if $POA = \phi = \frac{y}{r}$,

$$\begin{aligned} Pq &= Oq \frac{\sin \phi}{\sin \frac{\phi}{\mu}}, \\ &= Oq \frac{\phi - \frac{\phi^3}{6}}{\frac{\phi}{\mu} - \frac{\phi^3}{6\mu^3}}, \\ &= (v' - r) \frac{1 - \frac{y^2}{6r^2}}{\frac{1}{\mu} - \frac{y^2}{6\mu^3 r^2}}, \\ &= \mu (v' - r) \left(1 - \frac{y^2}{6r^2}\right) \left(1 + \frac{y^2}{6\mu^3 r^2}\right), \\ &= \mu (v' - r) \left(1 - \frac{\mu^2 - 1}{\mu^3} \frac{y^2}{6r^2}\right). \end{aligned}$$

But $Pq = v' \left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right) \frac{y^2}{2v'}\right\};$

therefore $v' - \left(\frac{1}{r} - \frac{1}{v'}\right) \frac{y^2}{2} = \mu (v' - r) \left\{1 - \frac{\mu^2 - 1}{\mu^3} \cdot \frac{y^2}{6r^2}\right\}.$

Hence $v = \mu (v - r),$

$$v = \frac{\mu r}{\mu - 1},$$

$$(\mu - 1) v' = \mu r + \mu (v' - r) \frac{\mu^2 - 1}{\mu^2} \cdot \frac{y^2}{6r^3} - \left(\frac{1}{r} - \frac{1}{v'} \right) \frac{y^2}{2},$$

$$v' = \frac{\mu r}{\mu - 1} + \frac{\mu + 1}{\mu} (v - r) \frac{y^2}{6r^3} - \frac{1}{\mu - 1} \left(\frac{1}{r} - \frac{1}{v} \right) \frac{y^2}{2};$$

$$\text{therefore } v' - v = \frac{\mu + 1}{\mu} \cdot \frac{r}{\mu - 1} \cdot \frac{y^2}{6r^3} - \frac{1}{\mu(\mu - 1)} \frac{y^2}{2r}.$$

But when the law of refraction is $\sin \phi = \mu \sin \phi'$, the aberration

$$= - \frac{1}{\mu(\mu - 1)} \cdot \frac{y^3}{2r};$$

$$\text{therefore the approximate error} = \frac{1}{6\mu} \cdot \frac{\mu + 1}{\mu - 1} \cdot \frac{y^3}{r}.$$

7. If the angle of incidence be to the angle of refraction as $\mu + 1 : \mu$, then the surface which will refract parallel rays accurately to a point S , is formed by the revolution of the curve $\rho = a(\cos \mu\theta)^{\frac{1}{\mu}}$ round an axis drawn through S parallel to the incident rays, ρ being measured from S , and θ the inclination of ρ to the axis.

Let QP (fig. 60) be a ray incident at the point P of the curve, and refracted in direction SP ; Pp the normal at P , Pt the tangent, and $SPp = \phi$, $PSp = \theta$.

$$\text{Then } QPp = \phi + \theta = \frac{\mu + 1}{\mu} \phi;$$

$$\text{therefore } \phi = \mu\theta,$$

$$\tan SPt = \frac{-rd\theta}{dr} = \cot \phi = \cot \mu\theta,$$

$$\frac{dr}{r} = -\tan \mu\theta d\theta = \frac{1}{\mu} \cdot \frac{d(\cos \mu\theta)}{\cos \mu\theta}.$$

Integrating, and supposing that $r = a$ when $\theta = 0$,

$$\log \frac{r}{a} = \frac{1}{\mu} \log \cos \mu\theta;$$

$$\text{therefore } r = a(\cos \mu\theta)^{\frac{1}{\mu}}.$$

Ex. 1. The distance between a luminous point and the center of the pupil of the eye is (D), and the radius of the pupil (r), which is supposed very small compared with D : shew that the illumination of the pupil is

$$\frac{C\pi r^2}{D^2} \cos a \left(1 - \frac{3}{4} \frac{r^2}{D^2} + \frac{15}{8} \frac{r^2}{D^2} \sin^2 a \right) \text{ nearly,}$$

where a is the angle which D makes with the axis of the eye, and higher powers of $\frac{r}{D}$ are neglected.

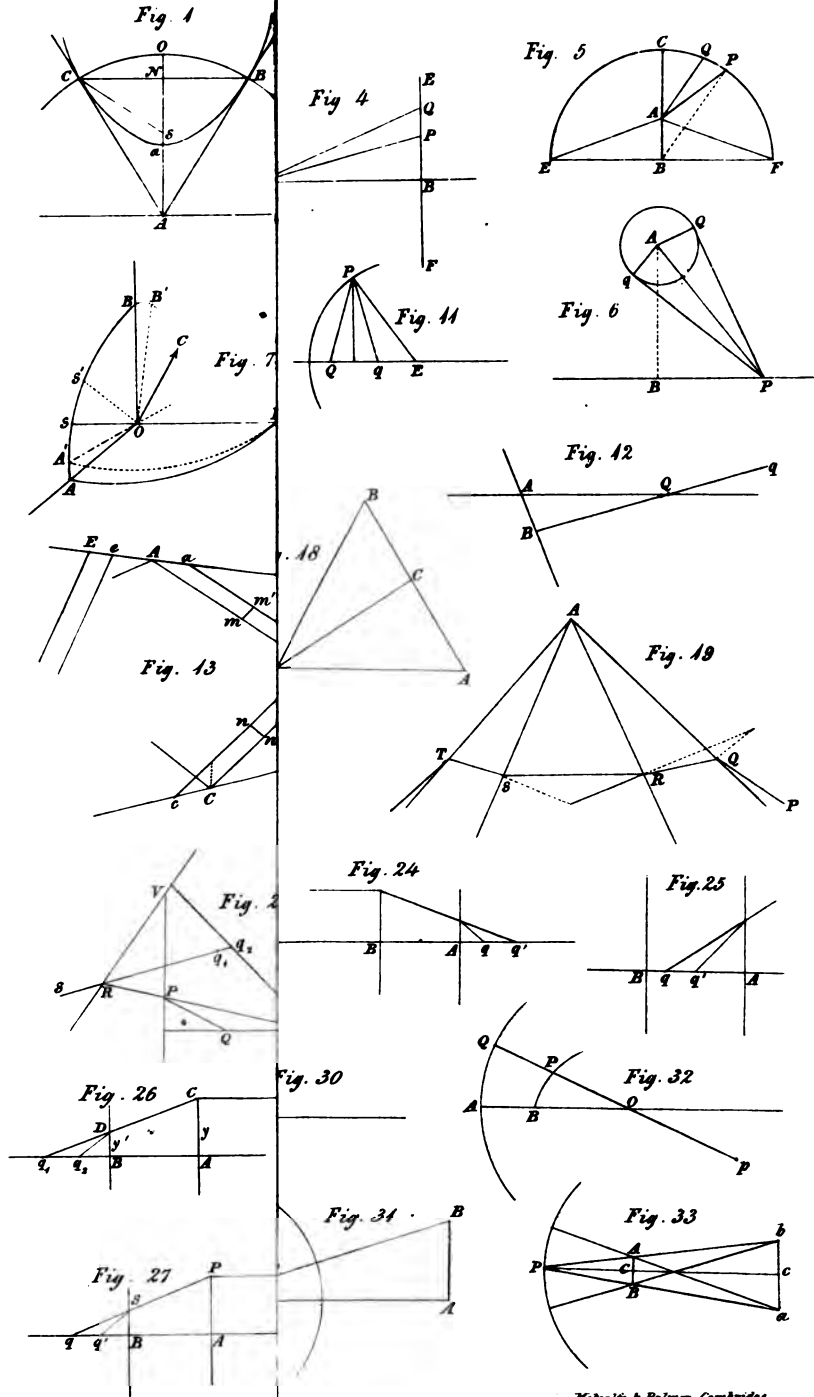
Ex. 2. A prolate spheroid, denser than the surrounding medium, has an origin of light in one of the foci of the generating ellipse, whose eccentricity = $\frac{1}{\sqrt{(\mu^2 - 1)}}$: shew that the whole of the rays will finally emerge, and determine the portion of the surface which will appear luminous.

Ex. 3. If two similar curves roll one upon another, similar points being originally in contact, and if a point upon one which is a reflecting curve be an origin of light, the caustic is the evolute of the curve traced out by the similarly situated point in the rolling curve.

Ex. 4. The extremity of the shadow of a vertical post standing on one side of a pond falls on a point A at the bottom of it. This point cannot be seen by a person on the opposite side, being just hidden by the sun's image in the water. Given the tangent of the sun's altitude to be $\frac{3}{4}$, the height of the post above the surface of the water 12 feet, that of the eye 18 feet, the breadth of the pond 70 feet, and the index of refraction out of air into water $\frac{4}{3}$: prove that the depth of the pond is 20 feet.

THE END.

Plate 1.



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Fig. 34 29 38

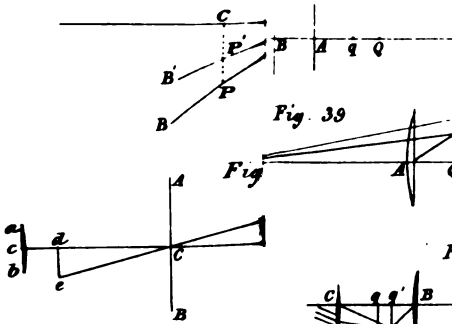


Fig. 39

Fig. 40

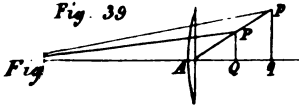


Fig. 40.

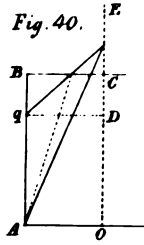


Fig. 44

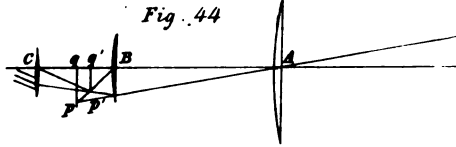


Fig. 47

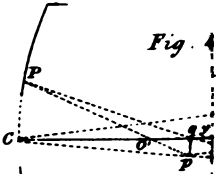
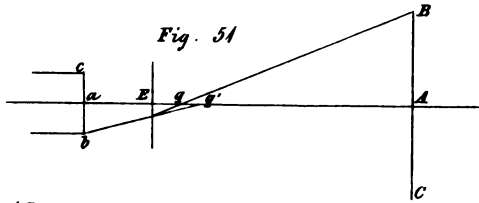


Fig. 51



47

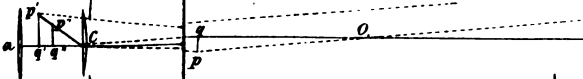


Fig. 52

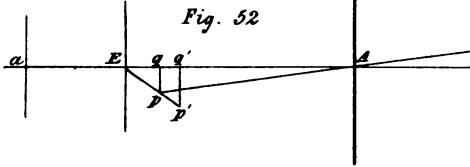


Fig. 55

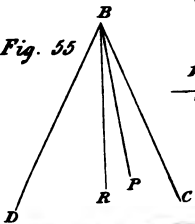


Fig. 56

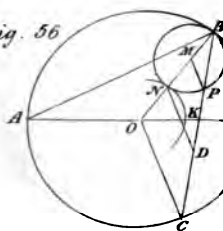


Fig. 57

